Shortlisted Problems

(with solutions)

Confidential until

1:30pm on 12 July 2022

(Norwegian time)

62nd International Mathematical Olympiad

Saint-Petersburg — Russia, 16th–24th July 2021

Note of Confidentiality

The Shortlist has to be kept strictly confidential until the conclusion of the following International Mathematical Olympiad.

IMO General Regulations §6.6

Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2021 thank the following 51 countries for contributing 175 problem proposals:

Albania, Algeria, Armenia, Australia, Austria, Azerbaijan, Belgium, Bangladesh, Canada, China, Colombia, Croatia, Czech Republic, Denmark, Estonia, France, Germany, Greece, Hong Kong, Iceland, India, Iran, Ireland, Israel, Japan, Kazakhstan, Kosovo, Luxembourg, Malaysia, Mexico, Morocco, Myanmar, Netherlands, New Zealand, North Macedonia, Poland, Romania, Singapore, Slovakia, Slovenia, South Africa, South Korea, Spain, Switzerland, Taiwan, Thailand, U.S.A., Ukraine, United Kingdom, Uzbekistan, Vietnam

Problem Selection Committee

Géza Kós, Gerhard Woeginger, Alexey Ustinov, Dmitry Krachun, Ivan Mitrofanov, Sergey Berlov, Fedor Petrov, Ivan Frolov, Paul Vaderlind, Alexander Golovanov, Ilya I. Bogdanov (chair)

Problems

Algebra

A1. Let n be an integer, and let A be a subset of $\{0, 1, 2, 3, \ldots, 5^n\}$ consisting of $4n+2$ numbers. Prove that there exist $a, b, c \in A$ such that $a < b < c$ and $c + 2a > 3b$.

A2. For every integer $n \geqslant 1$ consider the $n \times n$ table with entry $\left\lfloor \frac{ij}{n+1} \right\rfloor$ at the intersection of row i and column j, for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Determine all integers $n \geqslant 1$ for which the sum of the n^2 entries in the table is equal to $\frac{1}{4}n^2(n-1)$.

A3. Given a positive integer n, find the smallest value of $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor$ over all permutations $(a_1, a_2, \ldots, a_n)$ of $(1, 2, \ldots, n)$.

A4. Show that for all real numbers $x_1, \ldots, x_n$ the following inequality holds:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i + x_j|}.$$

A5. Let $n \geqslant 2$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1 - a_k}(a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

A6. Let A be a finite set of (not necessarily positive) integers, and let $m \geqslant 2$ be an integer. Assume that there exist non-empty subsets $B_1, B_2, B_3, \ldots, B_m$ of A whose elements add up to the sums $m^1, m^2, m^3, \ldots, m^m$, respectively. Prove that A contains at least $m/2$ elements.

A7. Let $n \geqslant 1$ be an integer, and let $x_0, x_1, \ldots, x_{n+1}$ be $n+2$ non-negative real numbers that satisfy $x_i x_{i+1} - x_{i-1}^2 \geqslant 1$ for all $i = 1, 2, \ldots, n$. Show that

$$x_0 + x_1 + \cdots + x_n + x_{n+1} > \left(\frac{2n}{3}\right)^{3/2}.$$

A8. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$\big(f(a) - f(b)\big)\big(f(b) - f(c)\big)\big(f(c) - f(a)\big) = f(ab^2 + bc^2 + ca^2) - f(a^2b + b^2c + c^2a)$$

for all real numbers a, b, c.

Combinatorics

C1. Let S be an infinite set of positive integers, such that there exist four pairwise distinct $a, b, c, d \in S$ with $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist three pairwise distinct $x, y, z \in S$ such that $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.

C2. Let $n \geqslant 3$ be an integer. An integer $m \geqslant n+1$ is called *n-colourful* if, given infinitely many marbles in each of n colours $C_1, C_2, \ldots, C_n$, it is possible to place m of them around a circle so that in any group of $n+1$ consecutive marbles there is at least one marble of colour C_i for each $i = 1, \ldots, n$.

Prove that there are only finitely many positive integers which are not n-colourful. Find the largest among them.

C3. A thimblerigger has 2021 thimbles numbered from 1 through 2021. The thimbles are arranged in a circle in arbitrary order. The thimblerigger performs a sequence of 2021 moves; in the k^{th} move, he swaps the positions of the two thimbles adjacent to thimble k.

Prove that there exists a value of k such that, in the k^{th} move, the thimblerigger swaps some thimbles a and b such that $a < k < b$.

C4. The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a *path from X to Y* is a sequence of roads such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called *diverse* if no road belongs to two or more paths in the collection.

Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

C5. Let n and k be two integers with $n > k \geqslant 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ *neighbours*— namely, the k students closest to S on the right, and the k students closest to S on the left.

Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbours.

C6. A hunter and an invisible rabbit play a game on an infinite square grid. First the hunter fixes a colouring of the cells with finitely many colours. The rabbit then secretly chooses a cell to start in. Every minute, the rabbit reports the colour of its current cell to the hunter, and then secretly moves to an adjacent cell that it has not visited before (two cells are adjacent if they share a side). The hunter wins if after some finite time either

- the rabbit cannot move; or
- the hunter can determine the cell in which the rabbit started.

Decide whether there exists a winning strategy for the hunter.

C7. Consider a checkered $3m \times 3m$ square, where m is an integer greater than 1. A frog sits on the lower left corner cell S and wants to get to the upper right corner cell F. The frog can hop from any cell to either the next cell to the right or the next cell upwards.

Some cells can be sticky, and the frog gets trapped once it hops on such a cell. A set X of cells is called *blocking* if the frog cannot reach F from S when all the cells of X are sticky. A blocking set is *minimal* if it does not contain a smaller blocking set.

(a) Prove that there exists a minimal blocking set containing at least $3m^2 - 3m$ cells.

(b) Prove that every minimal blocking set contains at most $3m^2$ cells.

Note. An example of a minimal blocking set for $m = 2$ is shown below. Cells of the set X are marked by letters x.

					F
x	x				
		x			
			x		
				x	
S		x			

C8. Determine the largest N for which there exists a table T of integers with N rows and 100 columns that has the following properties:

(i) Every row contains the numbers 1, 2, …, 100 in some order.

(ii) For any two distinct rows r and s, there is a column c such that $|T(r,c) - T(s,c)| \geqslant 2$.

Here $T(r,c)$ means the number at the intersection of the row r and the column c.

Geometry

G1. Let $ABCD$ be a parallelogram such that $AC = BC$. A point P is chosen on the extension of the segment AB beyond B. The circumcircle of the triangle ACD meets the segment PD again at Q, and the circumcircle of the triangle APQ meets the segment PC again at R. Prove that the lines CD, AQ, and BR are concurrent.

G2. Let $ABCD$ be a convex quadrilateral circumscribed around a circle with centre I. Let ω be the circumcircle of the triangle ACI. The extensions of BA and BC beyond A and C meet ω at X and Z, respectively. The extensions of AD and CD beyond D meet ω at Y and T, respectively. Prove that the perimeters of the (possibly self-intersecting) quadrilaterals $ADTX$ and $CDYZ$ are equal.

G3.

Version 1. Let n be a fixed positive integer, and let S be the set of points (x, y) on the Cartesian plane such that both coordinates x and y are nonnegative integers smaller than $2n$ (thus $|\mathsf{S}| = 4n^2$). Assume that $\mathcal{F}$ is a set consisting of n^2 quadrilaterals such that all their vertices lie in S, and each point in S is a vertex of exactly one of the quadrilaterals in $\mathcal{F}$.

Determine the largest possible sum of areas of all n^2 quadrilaterals in $\mathcal{F}$.

Version 2. Let n be a fixed positive integer, and let S be the set of points (x, y) on the Cartesian plane such that both coordinates x and y are nonnegative integers smaller than $2n$ (thus $|\mathsf{S}| = 4n^2$). Assume that $\mathcal{F}$ is a set of polygons such that all vertices of polygons in $\mathcal{F}$ lie in S, and each point in S is a vertex of exactly one of the polygons in $\mathcal{F}$.

Determine the largest possible sum of areas of all polygons in $\mathcal{F}$.

G4. Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D intersect the rays BA and BC at points E and F, respectively. A point T is chosen inside the triangle ABC so that $TE \parallel CD$ and $TF \parallel AD$. Let $K \neq D$ be a point on the segment DF such that $TD = TK$. Prove that the lines AC, DT and BK intersect at one point.

G5. Let $ABCD$ be a cyclic quadrilateral whose sides have pairwise different lengths. Let O be the circumcentre of $ABCD$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at B_1 and D_1, respectively. Let O_B be the centre of the circle which passes through B and is tangent to AC at D_1. Similarly, let O_D be the centre of the circle which passes through D and is tangent to AC at B_1.

Assume that $BD_1 \parallel DB_1$. Prove that O lies on the line O_BO_D.

G6. Determine all integers $n \geqslant 3$ satisfying the following property: every convex n-gon whose sides all have length 1 contains an equilateral triangle of side length 1.

(Every polygon is assumed to contain its boundary.)

G7. A point D is chosen inside an acute-angled triangle ABC with $AB > AC$ so that $\angle BAD = \angle DAC$. A point E is constructed on the segment AC so that $\angle ADE = \angle DCB$. Similarly, a point F is constructed on the segment AB so that $\angle ADF = \angle DBC$. A point X is chosen on the line AC so that $CX = BX$. Let O_1 and O_2 be the circumcentres of the triangles ADC and DXE. Prove that the lines BC, EF, and O_1O_2 are concurrent.

G8. Let ω be the circumcircle of a triangle ABC, and let Ω_A be its excircle which is tangent to the segment BC. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y, respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $AR \perp BC$.

Number Theory

N1. Determine all integers $n \geqslant 1$ for which there exists a pair of positive integers (a, b) such that no cube of a prime divides $a^2 + b + 3$ and

$$\frac{ab + 3b + 8}{a^2 + b + 3} = n.$$

N2. Let $n \geqslant 100$ be an integer. The numbers $n, n+1, \ldots, 2n$ are written on $n+1$ cards, one number per card. The cards are shuffled and divided into two piles. Prove that one of the piles contains two cards such that the sum of their numbers is a perfect square.

N3. Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \ldots, d_k)$ such that for every $i = 1, 2, \ldots, k$, the number $d_1 + \cdots + d_i$ is a perfect square.

N4. Alice is given a rational number $r > 1$ and a line with two points $B \neq R$, where point R contains a red bead and point B contains a blue bead. Alice plays a solitaire game by performing a sequence of moves. In every move, she chooses a (not necessarily positive) integer k, and a bead to move. If that bead is placed at point X, and the other bead is placed at Y, then Alice moves the chosen bead to point X' with $\overrightarrow{YX'} = r^k \overrightarrow{YX}$.

Alice's goal is to move the red bead to the point B. Find all rational numbers $r > 1$ such that Alice can reach her goal in at most 2021 moves.

N5. Prove that there are only finitely many quadruples (a, b, c, n) of positive integers such that

$$n! = a^{n-1} + b^{n-1} + c^{n-1}.$$

N6. Determine all integers $n \geqslant 2$ with the following property: every n pairwise distinct integers whose sum is not divisible by n can be arranged in some order $a_1, a_2, \ldots, a_n$ so that n divides $1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n$.

N7. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers such that a_{n+2m} divides $a_n + a_{n+m}$ for all positive integers n and m. Prove that this sequence is eventually periodic, i.e. there exist positive integers N and d such that $a_n = a_{n+d}$ for all $n > N$.

N8. For a polynomial $P(x)$ with integer coefficients let $P^1(x) = P(x)$ and $P^{k+1}(x) = P(P^k(x))$ for $k \geqslant 1$. Find all positive integers n for which there exists a polynomial $P(x)$ with integer coefficients such that for every integer $m \geqslant 1$, the numbers $P^m(1), \ldots, P^m(n)$ leave exactly $\lceil n/2^m \rceil$ distinct remainders when divided by n.

This page is intentionally left blank

Solutions

Algebra

A1. Let n be an integer, and let A be a subset of $\{0, 1, 2, 3, \ldots, 5^n\}$ consisting of $4n + 2$ numbers. Prove that there exist $a, b, c \in A$ such that $a < b < c$ and $c + 2a > 3b$.

Solution 1. (By contradiction) Suppose that there exist $4n + 2$ non-negative integers $x_0 < x_1 < \cdots < x_{4n+1}$ that violate the problem statement. Then in particular $x_{4n+1} + 2x_i \leqslant 3x_{i+1}$ for all $i = 0, \ldots, 4n - 1$, which gives

$$x_{4n+1} - x_i \geqslant \frac{3}{2}(x_{4n+1} - x_{i+1}).$$

By a trivial induction we then get

$$x_{4n+1} - x_i \geqslant \left(\frac{3}{2}\right)^{4n-i}(x_{4n+1} - x_{4n}),$$

which for $i = 0$ yields the contradiction

$$x_{4n+1} - x_0 \geqslant \left(\frac{3}{2}\right)^{4n}(x_{4n+1} - x_{4n}) = \left(\frac{81}{16}\right)^{n}(x_{4n+1} - x_{4n}) > 5^n \cdot 1.$$

Solution 2. Denote the maximum element of A by c. For $k = 0, \ldots, 4n - 1$ let

$$A_k = \left\{x \in A : (1 - (2/3)^k)c \leqslant x < (1 - (2/3)^{k+1})c\right\}.$$

Note that

$$(1 - (2/3)^{4n})c = c - (16/81)^n c > c - (1/5)^n c \geqslant c - 1,$$

which shows that the sets $A_0, A_1, \ldots, A_{4n-1}$ form a partition of $A \setminus \{c\}$. Since $A \setminus \{c\}$ has $4n+1$ elements, by the pigeonhole principle some set A_k does contain at least two elements of $A \setminus \{c\}$. Denote these two elements a and b and assume $a < b$, so that $a < b < c$. Then

$$c + 2a \geqslant c + 2(1 - (2/3)^k)c = (3 - 2(2/3)^k)c = 3(1 - (2/3)^{k+1})c > 3b,$$

as desired.

A2. For every integer $n \geqslant 1$ consider the $n \times n$ table with entry $\left\lfloor \frac{ij}{n+1} \right\rfloor$ at the intersection of row i and column j, for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Determine all integers $n \geqslant 1$ for which the sum of the n^2 entries in the table is equal to $\frac{1}{4}n^2(n-1)$.

Answer: All integers n for which $n+1$ is a prime.

Solution 1. First, observe that every pair x, y of real numbers for which the sum $x+y$ is integer satisfies

$$\lfloor x \rfloor + \lfloor y \rfloor \geqslant x + y - 1. \tag{1}$$

The inequality is strict if x and y are integers, and it holds with equality otherwise.

We estimate the sum S as follows.

$$\begin{aligned} 2S = \sum_{1 \leqslant i,j \leqslant n} \left(\left\lfloor \frac{ij}{n+1} \right\rfloor + \left\lfloor \frac{ij}{n+1} \right\rfloor \right) &= \sum_{1 \leqslant i,j \leqslant n} \left(\left\lfloor \frac{ij}{n+1} \right\rfloor + \left\lfloor \frac{(n+1-i)j}{n+1} \right\rfloor \right) \\ &\geqslant \sum_{1 \leqslant i,j \leqslant n} (j-1) = \frac{(n-1)n^2}{2}. \end{aligned}$$

The inequality in the last line follows from (1) by setting $x = ij/(n+1)$ and $y = (n+1-i)j/(n+1)$, so that $x + y = j$ is integral.

Now $S = \frac{1}{4}n^2(n-1)$ if and only if the inequality in the last line holds with equality, which means that none of the values $ij/(n+1)$ with $1 \leqslant i, j \leqslant n$ may be integral.

Hence, if $n+1$ is composite with factorisation $n+1 = ab$ for $2 \leqslant a, b \leqslant n$, one gets a strict inequality for $i = a$ and $j = b$. If $n+1$ is a prime, then $ij/(n+1)$ is never integral and $S = \frac{1}{4}n^2(n-1)$.

Solution 2. To simplify the calculation with indices, extend the table by adding a phantom column of index 0 with zero entries (which will not change the sum of the table). Fix a row i with $1 \leqslant i \leqslant n$, and let $d := \gcd(i, n+1)$ and $k := (n+1)/d$. For columns $j = 0, \ldots, n$, define the remainder $r_j := ij \bmod (n+1)$. We first prove the following

Claim. For every integer g with $1 \leqslant g \leqslant d$, the remainders r_j with indices j in the range

$$(g-1)k \leqslant j \leqslant gk - 1 \tag{2}$$

form a permutation of the k numbers $0 \cdot d$, $1 \cdot d$, $2 \cdot d$, $\ldots$, $(k-1) \cdot d$.

Proof. If $r_{j'} = r_j$ holds for two indices j' and j in (2), then $i(j'-j) \equiv 0 \bmod (n+1)$, so that $j'-j$ is a multiple of k; since $|j'-j| \leqslant k-1$, this implies $j' = j$. Hence, the k remainders are pairwise distinct. Moreover, each remainder $r_j = ij \bmod (n+1)$ is a multiple of $d = \gcd(i, n+1)$. This proves the claim.

We then have

$$\sum_{j=0}^{n} r_j = \sum_{g=1}^{d} \sum_{\ell=0}^{(n+1)/d-1} \ell d = d^2 \cdot \frac{1}{2} \left(\frac{n+1}{d} - 1 \right) \frac{n+1}{d} = \frac{(n+1-d)(n+1)}{2}. \tag{3}$$

By using (3), compute the sum S_i of row i as follows:

$$\begin{aligned} S_i &= \sum_{j=0}^{n} \left\lfloor \frac{ij}{n+1} \right\rfloor = \sum_{j=0}^{n} \frac{ij - r_j}{n+1} = \frac{i}{n+1} \sum_{j=0}^{n} j - \frac{1}{n+1} \sum_{j=0}^{n} r_j \\ &= \frac{i}{n+1} \cdot \frac{n(n+1)}{2} - \frac{1}{n+1} \cdot \frac{(n+1-d)(n+1)}{2} = \frac{(in - n - 1 + d)}{2}. \end{aligned} \tag{4}$$

Equation (4) yields the following lower bound on the row sum S_i, which holds with equality if and only if $d = \gcd(i, n+1) = 1$:

$$S_i \geqslant \frac{(in - n - 1 + 1)}{2} = \frac{n(i-1)}{2}. \tag{5}$$

By summing up the bounds (5) for the rows $i = 1, \ldots, n$, we get the following lower bound for the sum of all entries in the table

$$\sum_{i=1}^{n} S_i \geqslant \sum_{i=1}^{n} \frac{n}{2}(i-1) = \frac{n^2(n-1)}{4}. \tag{6}$$

In (6) we have equality if and only if equality holds in (5) for each $i = 1, \ldots, n$, which happens if and only if $\gcd(i, n+1) = 1$ for each $i = 1, \ldots, n$, which is equivalent to the fact that $n+1$ is a prime. Thus the sum of the table entries is $\frac{1}{4}n^2(n-1)$ if and only if $n+1$ is a prime.

Comment. To simplify the answer, in the problem statement one can make a change of variables by introducing $m := n+1$ and writing everything in terms of m. The drawback is that the expression for the sum will then be $\frac{1}{4}(m-1)^2(m-2)$ which seems more artificial.

A3. Given a positive integer n, find the smallest value of $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor$ over all permutations $(a_1, a_2, \ldots, a_n)$ of $(1, 2, \ldots, n)$.

Answer: The minimum of such sums is $\lfloor \log_2 n \rfloor + 1$; so if $2^k \leqslant n < 2^{k+1}$, the minimum is $k+1$.

Solution 1. Suppose that $2^k \leqslant n < 2^{k+1}$ with some nonnegative integer k. First we show a permutation $(a_1, a_2, \ldots, a_n)$ such that $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor = k+1$; then we will prove that $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor \geqslant k+1$ for every permutation. Hence, the minimal possible value will be $k+1$.

I. Consider the permutation

$$(a_1) = (1), \quad (a_2, a_3) = (3, 2), \quad (a_4, a_5, a_6, a_7) = (7, 4, 5, 6), \quad \ldots$$
$$(a_{2^{k-1}}, \ldots, a_{2^k-1}) = (2^k - 1, 2^{k-1}, 2^{k-1} + 1, \ldots, 2^k - 2),$$
$$(a_{2^k}, \ldots, a_n) = (n, 2^k, 2^k + 1, \ldots, n-1).$$

This permutation consists of $k+1$ cycles. In every cycle $(a_p, \ldots, a_q) = (q, p, p+1, \ldots, q-1)$ we have $q < 2p$, so

$$\sum_{i=p}^{q} \left\lfloor \frac{a_i}{i} \right\rfloor = \left\lfloor \frac{q}{p} \right\rfloor + \sum_{i=p+1}^{q} \left\lfloor \frac{i-1}{i} \right\rfloor = 1;$$

The total sum over all cycles is precisely $k+1$.

II. In order to establish the lower bound, we prove a more general statement.

Claim. If $b_1, \ldots, b_{2^k}$ are distinct positive integers then

$$\sum_{i=1}^{2^k} \left\lfloor \frac{b_i}{i} \right\rfloor \geqslant k+1.$$

From the Claim it follows immediately that $\sum_{i=1}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \sum_{i=1}^{2^k} \left\lfloor \frac{a_i}{i} \right\rfloor \geqslant k+1$.

Proof of the Claim. Apply induction on k. For $k=1$ the claim is trivial, $\left\lfloor \frac{b_1}{1} \right\rfloor \geqslant 1$. Suppose the Claim holds true for some positive integer k, and consider $k+1$.

If there exists an index j such that $2^k < j \leqslant 2^{k+1}$ and $b_j \geqslant j$ then

$$\sum_{i=1}^{2^{k+1}} \left\lfloor \frac{b_i}{i} \right\rfloor \geqslant \sum_{i=1}^{2^k} \left\lfloor \frac{b_i}{i} \right\rfloor + \left\lfloor \frac{b_j}{j} \right\rfloor \geqslant (k+1) + 1$$

by the induction hypothesis, so the Claim is satisfied.

Otherwise we have $b_j < j \leqslant 2^{k+1}$ for every $2^k < j \leqslant 2^{k+1}$. Among the 2^{k+1} distinct numbers $b_1, \ldots, b_{2^{k+1}}$ there is some b_m which is at least 2^{k+1}; that number must be among $b_1 \ldots, b_{2^k}$. Hence, $1 \leqslant m \leqslant 2^k$ and $b_m \geqslant 2^{k+1}$.

We will apply the induction hypothesis to the numbers

$$c_1 = b_1, \ldots, c_{m-1} = b_{m-1}, \quad c_m = b_{2^k+1}, \quad c_{m+1} = b_{m+1}, \ldots, c_{2^k} = b_{2^k},$$

so take the first 2^k numbers but replace b_m with b_{2^k+1}. Notice that

$$\left\lfloor \frac{b_m}{m} \right\rfloor \geqslant \left\lfloor \frac{2^{k+1}}{m} \right\rfloor = \left\lfloor \frac{2^k + 2^k}{m} \right\rfloor \geqslant \left\lfloor \frac{b_{2^k+1} + m}{m} \right\rfloor = \left\lfloor \frac{c_m}{m} \right\rfloor + 1.$$

For the other indices i with $1 \leqslant i \leqslant 2^k$, $i \neq m$ we have $\left\lfloor \frac{b_i}{i} \right\rfloor = \left\lfloor \frac{c_i}{i} \right\rfloor$, so

$$\sum_{i=1}^{2^{k+1}} \left\lfloor \frac{b_i}{i} \right\rfloor = \sum_{i=1}^{2^k} \left\lfloor \frac{b_i}{i} \right\rfloor \geqslant \sum_{i=1}^{2^k} \left\lfloor \frac{c_i}{i} \right\rfloor + 1 \geqslant (k+1) + 1.$$

That proves the Claim and hence completes the solution. $\square$

Solution 2. We present a different proof for the lower bound.

Assume again $2^k \leqslant n < 2^{k+1}$, and let $P = \{2^0, 2^1, \ldots, 2^k\}$ be the set of powers of 2 among $1, 2, \ldots, n$. Call an integer $i \in \{1, 2, \ldots, n\}$ and the interval $[i, a_i]$ *good* if $a_i \geqslant i$.

Lemma 1. The good intervals cover the integers $1, 2, \ldots, n$.

Proof. Consider an arbitrary $x \in \{1, 2 \ldots, n\}$; we want to find a good interval $[i, a_i]$ that covers x; i.e., $i \leqslant x \leqslant a_i$. Take the cycle of the permutation that contains x, that is $(x, a_x, a_{a_x}, \ldots)$. In this cycle, let i be the first element with $a_i \geqslant x$; then $i \leqslant x \leqslant a_i$. $\square$

Lemma 2. If a good interval $[i, a_i]$ covers p distinct powers of 2 then $\left\lfloor \frac{a_i}{i} \right\rfloor \geqslant p$; more formally, $\left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \left| [i, a_i] \cap P \right|$.

Proof. The ratio of the smallest and largest powers of 2 in the interval is at least 2^{p-1}. By Bernoulli's inequality, $\frac{a_i}{i} \geqslant 2^{p-1} \geqslant p$; that proves the lemma. $\square$

Now, by Lemma 1, the good intervals cover P. By applying Lemma 2 as well, we obtain that

$$\sum_{i=1}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor = \sum_{i \text{ is good}}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \sum_{i \text{ is good}}^{n} \left| [i, a_i] \cap P \right| \geqslant |P| = k+1.$$

Solution 3. We show yet another proof for the lower bound, based on the following inequality.

Lemma 3.

$$\left\lfloor \frac{a}{b} \right\rfloor \geqslant \log_2 \frac{a+1}{b}$$

for every pair a, b of positive integers.

Proof. Let $t = \left\lfloor \frac{a}{b} \right\rfloor$, so $t \leqslant \frac{a}{b}$ and $\frac{a+1}{b} \leqslant t+1$. By applying the inequality $2^t \geqslant t+1$, we obtain

$$\left\lfloor \frac{a}{b} \right\rfloor = t \geqslant \log_2(t+1) \geqslant \log_2 \frac{a+1}{b}. \qquad \square$$

By applying the lemma to each term, we get

$$\sum_{i=1}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \sum_{i=1}^{n} \log_2 \frac{a_i+1}{i} = \sum_{i=1}^{n} \log_2(a_i+1) - \sum_{i=1}^{n} \log_2 i.$$

Notice that the numbers $a_1+1, a_2+1, \ldots, a_n+1$ form a permutation of $2, 3, \ldots, n+1$. Hence, in the last two sums all terms cancel out, except for $\log_2(n+1)$ in the first sum and $\log_2 1 = 0$ in the second sum. Therefore,

$$\sum_{i=1}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \log_2(n+1) > k.$$

As the left-hand side is an integer, it must be at least $k+1$.

A4. Show that for all real numbers $x_1, \ldots, x_n$ the following inequality holds:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}.$$

Solution 1. If we add t to all the variables then the left-hand side remains constant and the right-hand side becomes

$$H(t) := \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j + 2t|}.$$

Let T be large enough such that both $H(-T)$ and $H(T)$ are larger than the value L of the left-hand side of the inequality we want to prove. Not necessarily distinct points $p_{i,j} := -(x_i + x_j)/2$ together with T and $-T$ split the real line into segments and two rays such that on each of these segments and rays the function $H(t)$ is concave since $f(t) := \sqrt{|\ell + 2t|}$ is concave on both intervals $(-\infty, -\ell/2]$ and $[-\ell/2, +\infty)$. Let $[a, b]$ be the segment containing zero. Then concavity implies $H(0) \geqslant \min\{H(a), H(b)\}$ and, since $H(\pm T) > L$, it suffices to prove the inequalities $H(-(x_i + x_j)/2) \geqslant L$, that is to prove the original inequality in the case when all numbers are shifted in such a way that two variables x_i and x_j add up to zero. In the following we denote the shifted variables still by x_i.

If $i = j$, i.e. $x_i = 0$ for some index i, then we can remove x_i which will decrease both sides by $2\sum_k \sqrt{|x_k|}$. Similarly, if $x_i + x_j = 0$ for distinct i and j we can remove both x_i and x_j which decreases both sides by

$$2\sqrt{2|x_i|} + 2 \cdot \sum_{k \neq i,j}\left(\sqrt{|x_k + x_i|} + \sqrt{|x_k + x_j|}\right).$$

In either case we reduced our inequality to the case of smaller n. It remains to note that for $n = 0$ and $n = 1$ the inequality is trivial.

Solution 2. For real p consider the integral

$$I(p) = \int_0^\infty \frac{1 - \cos(px)}{x\sqrt{x}}\,dx,$$

which clearly converges to a strictly positive number. By changing the variable $y = |p|x$ one notices that $I(p) = \sqrt{|p|}I(1)$. Hence, by using the trigonometric formula $\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta$ we obtain

$$\sqrt{|a+b|} - \sqrt{|a-b|} = \frac{1}{I(1)}\int_0^\infty \frac{\cos((a-b)x) - \cos((a+b)x)}{x\sqrt{x}}\,dx = \frac{1}{I(1)}\int_0^\infty \frac{2\sin(ax)\sin(bx)}{x\sqrt{x}}\,dx,$$

from which our inequality immediately follows:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|} - \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} = \frac{2}{I(1)}\int_0^\infty \frac{\left(\sum_{i=1}^n \sin(x_i x)\right)^2}{x\sqrt{x}}\,dx \geqslant 0.$$

Comment 1. A more general inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|^r \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|x_i + x_j|^r$$

holds for any $r \in [0, 2]$. The first solution can be repeated verbatim for any $r \in [0, 1]$ but not for $r > 1$. In the second solution, by putting x^{r+1} in the denominator in place of $x\sqrt{x}$ we can prove the inequality for any $r \in (0, 2)$ and the cases $r = 0, 2$ are easy to check by hand.

Comment 2. In fact, the integral from Solution 2 can be computed explicitly, we have $I(1) = \sqrt{2\pi}$.

A5. Let $n \geqslant 2$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1-a_k}(a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

Solution 1. For all $k \leqslant n$, let

$$s_k = a_1 + a_2 + \cdots + a_k \qquad \text{and} \qquad b_k = \frac{a_k s_{k-1}^2}{1-a_k},$$

with the convention that $s_0 = 0$. Note that b_k is exactly a summand in the sum we need to estimate. We shall prove the inequality

$$b_k < \frac{s_k^3 - s_{k-1}^3}{3}. \tag{1}$$

Indeed, it suffices to check that

$$\begin{aligned}
(1) \iff\ & 0 < (1-a_k)\left((s_{k-1}+a_k)^3 - s_{k-1}^3\right) - 3a_k s_{k-1}^2 \\
\iff\ & 0 < (1-a_k)\left(3s_{k-1}^2 + 3s_{k-1}a_k + a_k^2\right) - 3s_{k-1}^2 \\
\iff\ & 0 < -3a_k s_{k-1}^2 + 3(1-a_k)s_{k-1}a_k + (1-a_k)a_k^2 \\
\iff\ & 0 < 3(1-a_k-s_{k-1})s_{k-1}a_k + (1-a_k)a_k^2
\end{aligned}$$

which holds since $a_k + s_{k-1} = s_k \leqslant 1$ and $a_k \in (0,1)$.

Thus, adding inequalities (1) for $k = 1, \ldots, n$, we conclude that

$$b_1 + b_2 + \cdots + b_n < \frac{s_n^3 - s_0^3}{3} = \frac{1}{3},$$

as desired.

Comment 1. There are many ways of proving (1) which can be written as

$$\frac{as^2}{1-a} - \frac{(a+s)^3 - s^3}{3} < 0, \tag{2}$$

for non-negative a and s satisfying $a + s \leqslant 1$ and $a > 0$.

E.g., note that for any fixed a the expression in (2) is quadratic in s with the leading coefficient $a/(1-a) - a > 0$. Hence, it is convex as a function in s, so it suffices to check the inequality at $s = 0$ and $s = 1 - a$. The former case is trivial and in the latter case the inequality can be rewritten as

$$as - \frac{3as(a+s) + a^3}{3} < 0,$$

which is trivial since $a + s = 1$.

Solution 2. First, let us define

$$S(a_1, \ldots, a_n) := \sum_{k=1}^{n} \frac{a_k}{1-a_k}(a_1 + a_2 + \cdots + a_{k-1})^2.$$

For some index i, denote $a_1 + \cdots + a_{i-1}$ by s. If we replace a_i with two numbers $a_i/2$ and $a_i/2$, i.e. replace the tuple $(a_1, \ldots, a_n)$ with $(a_1, \ldots, a_{i-1}, a_i/2, a_i/2, a_{i+1}, \ldots, a_n)$, the sum will increase by

$$\begin{aligned} S(a_1, \ldots, a_i/2, a_i/2, \ldots, a_n) - S(a_1, \ldots, a_n) &= \frac{a_i/2}{1 - a_i/2}\left(s^2 + (s + a_i/2)^2\right) - \frac{a_i}{1 - a_i}s^2 \\ &= a_i \frac{(1 - a_i)(2s^2 + sa_i + a_i^2/4) - (2 - a_i)s^2}{(2 - a_i)(1 - a_i)} \\ &= a_i \frac{(1 - a_i - s)sa_i + (1 - a_i)a_i^2/4}{(2 - a_i)(1 - a_i)}, \end{aligned}$$

which is strictly positive. So every such replacement strictly increases the sum. By repeating this process and making maximal number in the tuple tend to zero, we keep increasing the sum which will converge to

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$

This completes the proof.

Solution 3. We sketch a probabilistic version of the first solution. Let x_1, x_2, x_3 be drawn uniformly and independently at random from the segment $[0, 1]$. Let $I_1 \cup I_2 \cup \cdots \cup I_n$ be a partition of $[0, 1]$ into segments of length $a_1, a_2, \ldots, a_n$ in this order. Let $J_k := I_1 \cup \cdots \cup I_{k-1}$ for $k \geqslant 2$ and $J_1 := \varnothing$. Then

$$\begin{aligned} \frac{1}{3} &= \sum_{k=1}^{n} \mathbb{P}\{x_1 \geqslant x_2, x_3;\ x_1 \in I_k\} \\ &= \sum_{k=1}^{n} \Big(\mathbb{P}\{x_1 \in I_k;\ x_2, x_3 \in J_k\} + 2 \cdot \mathbb{P}\{x_1 \geqslant x_2;\ x_1, x_2 \in I_k;\ x_3 \in J_k\} \\ &\qquad\qquad + \mathbb{P}\{x_1 \geqslant x_2, x_3;\ x_1, x_2, x_3 \in I_k\}\Big) \\ &= \sum_{k=1}^{n} \left(a_k(a_1 + \cdots + a_{k-1})^2 + 2 \cdot \frac{a_k^2}{2} \cdot (a_1 + \cdots + a_{k-1}) + \frac{a_k^3}{3}\right) \\ &> \sum_{k=1}^{n} \left(a_k(a_1 + \cdots + a_{k-1})^2 + a_k^2(a_1 + \cdots + a_{k-1}) \cdot \frac{a_1 + \cdots + a_{k-1}}{1 - a_k}\right), \end{aligned}$$

where for the last inequality we used that $1 - a_k \geqslant a_1 + \cdots + a_{k-1}$. This completes the proof since

$$a_k + \frac{a_k^2}{1 - a_k} = \frac{a_k}{1 - a_k}.$$

A6. Let A be a finite set of (not necessarily positive) integers, and let $m \geqslant 2$ be an integer. Assume that there exist non-empty subsets $B_1, B_2, B_3, \dots, B_m$ of A whose elements add up to the sums $m^1, m^2, m^3, \dots, m^m$, respectively. Prove that A contains at least $m/2$ elements.

Solution. Let $A = \{a_1, \dots, a_k\}$. Assume that, on the contrary, $k = |A| < m/2$. Let

$$s_i := \sum_{j: a_j \in B_i} a_j$$

be the sum of elements of B_i. We are given that $s_i = m^i$ for $i = 1, \dots, m$.

Now consider all m^m expressions of the form

$$f(c_1, \dots, c_m) := c_1 s_1 + c_2 s_2 + \ldots + c_m s_m, \ c_i \in \{0, 1, \dots, m-1\} \text{ for all } i = 1, 2, \dots, m.$$

Note that every number $f(c_1, \dots, c_m)$ has the form

$$\alpha_1 a_1 + \ldots + \alpha_k a_k, \ \alpha_i \in \{0, 1, \dots, m(m-1)\}.$$

Hence, there are at most $(m(m-1)+1)^k < m^{2k} < m^m$ distinct values of our expressions; therefore, at least two of them coincide.

Since $s_i = m^i$, this contradicts the uniqueness of representation of positive integers in the base-m system.

Comment 1. For other rapidly increasing sequences of sums of B_i's the similar argument also provides lower estimates on $k = |A|$. For example, if the sums of B_i are equal to $1!$, $2!$, $3!$, $\dots$, $m!$, then for any fixed $\varepsilon > 0$ and large enough m we get $k \geqslant (1/2 - \varepsilon)m$. The proof uses the fact that the combinations $\sum c_i i!$ with $c_i \in \{0, 1, \dots, i\}$ are all distinct.

Comment 2. The problem statement holds also if A is a set of real numbers (not necessarily integers), the above proofs work in the real case.

A7. Let $n \geqslant 1$ be an integer, and let $x_0, x_1, \ldots, x_{n+1}$ be $n+2$ non-negative real numbers that satisfy $x_i x_{i+1} - x_{i-1}^2 \geqslant 1$ for all $i = 1, 2, \ldots, n$. Show that

$$x_0 + x_1 + \cdots + x_n + x_{n+1} > \left(\frac{2n}{3}\right)^{3/2}.$$

Solution 1.

Lemma 1.1. If a, b, c are non-negative numbers such that $ab - c^2 \geqslant 1$, then

$$(a+2b)^2 \geqslant (b+2c)^2 + 6.$$

Proof. $(a+2b)^2 - (b+2c)^2 = (a-b)^2 + 2(b-c)^2 + 6(ab-c^2) \geqslant 6.$ □

Lemma 1.2. $\sqrt{1} + \cdots + \sqrt{n} > \frac{2}{3} n^{3/2}$.

Proof. Bernoulli's inequality $(1+t)^{3/2} > 1 + \frac{3}{2}t$ for $0 > t \geqslant -1$ (or, alternatively, a straightforward check) gives

$$(k-1)^{3/2} = k^{3/2}\left(1 - \frac{1}{k}\right)^{3/2} > k^{3/2}\left(1 - \frac{3}{2k}\right) = k^{3/2} - \frac{3}{2}\sqrt{k}. \qquad (*)$$

Summing up $(*)$ over $k = 1, 2, \ldots, n$ yields

$$0 > n^{3/2} - \frac{3}{2}\left(\sqrt{1} + \cdots + \sqrt{n}\right).$$ □

Now put $y_i := 2x_i + x_{i+1}$ for $i = 0, 1, \ldots, n$. We get $y_0 \geqslant 0$ and $y_i^2 \geqslant y_{i-1}^2 + 6$ for $i = 1, 2, \ldots, n$ by Lemma 1.1. Thus, an easy induction on i gives $y_i \geqslant \sqrt{6i}$. Using this estimate and Lemma 1.2 we get

$$3(x_0 + \ldots + x_{n+1}) \geqslant y_1 + \ldots + y_n \geqslant \sqrt{6}\left(\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n}\right) > \sqrt{6} \cdot \frac{2}{3} n^{3/2} = 3\left(\frac{2n}{3}\right)^{3/2}.$$

Solution 2. Say that an index $i \in \{0, 1, \ldots, n+1\}$ is *good*, if $x_i \geqslant \sqrt{\frac{2}{3}i}$, otherwise call the index i *bad*.

Lemma 2.1. There are no two consecutive bad indices.

Proof. Assume the contrary and consider two bad indices $j, j+1$ with minimal possible j. Since 0 is good, we get $j > 0$, thus by minimality $j-1$ is a good index and we have

$$\frac{2}{3}\sqrt{j(j+1)} > x_j x_{j+1} \geqslant x_{j-1}^2 + 1 \geqslant \frac{2}{3}(j-1) + 1 = \frac{2}{3} \cdot \frac{j + (j+1)}{2}$$

that contradicts the AM–GM inequality for numbers j and $j+1$. □

Lemma 2.2. If an index $j \leqslant n-1$ is good, then

$$x_{j+1} + x_{j+2} \geqslant \sqrt{\frac{2}{3}}\left(\sqrt{j+1} + \sqrt{j+2}\right).$$

Proof. We have

$$x_{j+1} + x_{j+2} \geqslant 2\sqrt{x_{j+1}x_{j+2}} \geqslant 2\sqrt{x_j^2 + 1} \geqslant 2\sqrt{\frac{2}{3}j + 1} \geqslant \sqrt{\frac{2}{3}j + \frac{2}{3}} + \sqrt{\frac{2}{3}j + \frac{4}{3}},$$

the last inequality follows from concavity of the square root function, or, alternatively, from the AM–QM inequality for the numbers $\sqrt{\frac{2}{3}j + \frac{2}{3}}$ and $\sqrt{\frac{2}{3}j + \frac{4}{3}}$. □

Let $S_i = x_1 + \ldots + x_i$ and $T_i = \sqrt{\frac{2}{3}}(\sqrt{1} + \ldots + \sqrt{i})$.

Lemma 2.3. If an index i is good, then $S_i \geqslant T_i$.

Proof. Induction on i. The base case $i = 0$ is clear. Assume that the claim holds for good indices less than i and prove it for a good index $i > 0$.

If $i-1$ is good, then by the inductive hypothesis we get $S_i = S_{i-1} + x_i \geqslant T_{i-1} + \sqrt{\frac{2}{3}}i = T_i$.

If $i-1$ is bad, then $i > 1$, and $i-2$ is good by Lemma 2.1. Then using Lemma 2.2 and the inductive hypothesis we get

$$S_i = S_{i-2} + x_{i-1} + x_i \geqslant T_{i-2} + \sqrt{\frac{2}{3}}\left(\sqrt{i-1} + \sqrt{i}\right) = T_i. \qquad \square$$

Since either n or $n+1$ is good by Lemma 2.1, Lemma 2.3 yields in both cases $S_{n+1} \geqslant T_n$, and it remains to apply Lemma 1.2 from Solution 1.

Comment 1. Another way to get $(*)$ is the integral bound

$$k^{3/2} - (k-1)^{3/2} = \int_{k-1}^{k} \frac{3}{2}\sqrt{x}\, dx < \frac{3}{2}\sqrt{k}.$$

Comment 2. If $x_i = \sqrt{2/3} \cdot (\sqrt{i} + 1)$, the conditions of the problem hold. Indeed, the inequality to check is

$$(\sqrt{i}+1)(\sqrt{i+1}+1) - (\sqrt{i-1}+1)^2 \geqslant 3/2,$$

that rewrites as

$$\sqrt{i} + \sqrt{i+1} - 2\sqrt{i-1} \geqslant (i + 1/2) - \sqrt{i(i+1)} = \frac{1/4}{i + 1/2 + \sqrt{i(i+1)}},$$

which follows from

$$\sqrt{i} - \sqrt{i-1} = \frac{1}{\sqrt{i} + \sqrt{i-1}} > \frac{1}{2i}.$$

For these numbers we have $x_0 + \ldots + x_{n+1} = \left(\frac{2n}{3}\right)^{3/2} + O(n)$, thus the multiplicative constant $(2/3)^{3/2}$ in the problem statement is sharp.

A8. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$\big(f(a)-f(b)\big)\big(f(b)-f(c)\big)\big(f(c)-f(a)\big)=f(ab^2+bc^2+ca^2)-f(a^2b+b^2c+c^2a)$$

for all real numbers a, b, c.

Answer: $f(x) = \alpha x + \beta$ or $f(x) = \alpha x^3 + \beta$ where $\alpha \in \{-1, 0, 1\}$ and $\beta \in \mathbb{R}$.

Solution. It is straightforward to check that above functions satisfy the equation. Now let $f(x)$ satisfy the equation, which we denote $E(a, b, c)$. Then clearly $f(x) + C$ also does; therefore, we may suppose without loss of generality that $f(0) = 0$.We start with proving

Lemma. Either $f(x) \equiv 0$ or f is injective.

Proof. Denote by $\Theta \subseteq \mathbb{R}^2$ the set of points (a, b) for which $f(a) = f(b)$. Let $\Theta^* = \{(x, y) \in \Theta : x \neq y\}$. The idea is that if $(a, b) \in \Theta$, then by $E(a, b, x)$ we get

$$H_{a,b}(x) := (ab^2 + bx^2 + xa^2, a^2b + b^2x + x^2a) \in \Theta$$

for all real x. Reproducing this argument starting with $(a, b) \in \Theta^*$, we get more and more points in Θ. There are many ways to fill in the details, we give below only one of them.

Assume that $(a, b) \in \Theta^*$. Note that

$$g_-(x) := (ab^2 + bx^2 + xa^2) - (a^2b + b^2x + x^2a) = (a - b)(b - x)(x - a)$$

and

$$g_+(x) := (ab^2 + bx^2 + xa^2) + (a^2b + b^2x + x^2a) = (x^2 + ab)(a + b) + x(a^2 + b^2).$$

Hence, there exists x for which both $g_-(x) \neq 0$ and $g_+(x) \neq 0$. This gives a point $(\alpha, \beta) = H_{a,b}(x) \in \Theta^*$ for which $\alpha \neq -\beta$. Now compare $E(\alpha, 1, 0)$ and $E(\beta, 1, 0)$. The left-hand side expressions coincide, on right-hand side we get $f(\alpha) - f(\alpha^2) = f(\beta) - f(\beta^2)$, respectively. Hence, $f(\alpha^2) = f(\beta^2)$ and we get a point $(\alpha_1, \beta_1) := (\alpha^2, \beta^2) \in \Theta^*$ with both coordinates α_1, β_1 non-negative. Continuing squaring the coordinates, we get a point $(\gamma, \delta) \in \Theta^*$ for which $\delta > 5\gamma \geqslant 0$. Our nearest goal is to get a point $(0, r) \in \Theta^*$. If $\gamma = 0$, this is already done. If $\gamma > 0$, denote by x a real root of the quadratic equation $\delta\gamma^2 + \gamma x^2 + x\delta^2 = 0$, which exists since the discriminant $\delta^4 - 4\delta\gamma^3$ is positive. Also $x < 0$ since this equation cannot have non-negative root. For the point $H_{\delta,\gamma}(x) =: (0, r) \in \Theta$ the first coordinate is 0. The difference of coordinates equals $-r = (\delta - \gamma)(\gamma - x)(x - \delta) < 0$, so $r \neq 0$ as desired.

Now, let $(0, r) \in \Theta^*$. We get $H_{0,r}(x) = (rx^2, r^2x) \in \Theta$. Thus $f(rx^2) = f(r^2x)$ for all $x \in \mathbb{R}$. Replacing x to $-x$ we get $f(rx^2) = f(r^2x) = f(-r^2x)$, so f is even: $(a, -a) \in \Theta$ for all a. Then $H_{a,-a}(x) = (a^3 - ax^2 + xa^2, -a^3 + a^2x + x^2a) \in \Theta$ for all real a, x. Putting $x = \frac{1+\sqrt{5}}{2}a$ we obtain $(0, (1 + \sqrt{5})a^3) \in \Theta$ which means that $f(y) = f(0) = 0$ for every real y. $\square$

Hereafter we assume that f is injective and $f(0) = 0$. By $E(a, b, 0)$ we get

$$f(a)f(b)(f(a) - f(b)) = f(a^2b) - f(ab^2). \qquad (\heartsuit)$$

Let $\kappa := f(1)$ and note that $\kappa = f(1) \neq f(0) = 0$ by injectivity. Putting $b = 1$ in $(\heartsuit)$ we get

$$\kappa f(a)(f(a) - \kappa) = f(a^2) - f(a). \qquad (\clubsuit)$$

Subtracting the same equality for $-a$ we get

$$\kappa(f(a) - f(-a))(f(a) + f(-a) - \kappa) = f(-a) - f(a).$$

Now, if $a \neq 0$, by injectivity we get $f(a) - f(-a) \neq 0$ and thus

$$f(a) + f(-a) = \kappa - \kappa^{-1} =: \lambda. \qquad (\spadesuit)$$

It follows that

$$f(a) - f(b) = f(-b) - f(-a)$$

for all non-zero a, b. Replace non-zero numbers a, b in ($\heartsuit$) with $-a, -b$, respectively, and add the two equalities. Due to ($\spadesuit$) we get

$$(f(a) - f(b))(f(a)f(b) - f(-a)f(-b)) = 0,$$

thus $f(a)f(b) = f(-a)f(-b) = (\lambda - f(a))(\lambda - f(b))$ for all non-zero $a \neq b$. If $\lambda \neq 0$, this implies $f(a) + f(b) = \lambda$ that contradicts injectivity when we vary b with fixed a. Therefore, $\lambda = 0$ and $\kappa = \pm 1$. Thus f is odd. Replacing f with $-f$ if necessary (this preserves the original equation) we may suppose that $f(1) = 1$.

Now, ($\clubsuit$) yields $f(a^2) = f^2(a)$. Summing relations ($\heartsuit$) for pairs (a, b) and $(a, -b)$, we get $-2f(a)f^2(b) = -2f(ab^2)$, i.e. $f(a)f(b^2) = f(ab^2)$. Putting $b = \sqrt{x}$ for each non-negative x we get $f(ax) = f(a)f(x)$ for all real a and non-negative x. Since f is odd, this multiplicativity relation is true for all a, x. Also, from $f(a^2) = f^2(a)$ we see that $f(x) \geqslant 0$ for $x \geqslant 0$. Next, $f(x) > 0$ for $x > 0$ by injectivity.

Assume that $f(x)$ for $x > 0$ does not have the form $f(x) = x^\tau$ for a constant τ. The known property of multiplicative functions yields that the graph of f is dense on $(0, \infty)^2$. In particular, we may find positive $b < 1/10$ for which $f(b) > 1$. Also, such b can be found if $f(x) = x^\tau$ for some $\tau < 0$. Then for all x we have $x^2 + xb^2 + b \geqslant 0$ and so $E(1, b, x)$ implies that

$$f(b^2 + bx^2 + x) = f(x^2 + xb^2 + b) + (f(b) - 1)(f(x) - f(b))(f(x) - 1) \geqslant 0 - ((f(b) - 1)^3/4$$

is bounded from below (the quadratic trinomial bound $(t - f(1))(t - f(b)) \geqslant -(f(b) - 1)^2/4$ for $t = f(x)$ is used). Hence, f is bounded from below on $(b^2 - \frac{1}{4b}, +\infty)$, and since f is odd it is bounded from above on $(0, \frac{1}{4b} - b^2)$. This is absurd if $f(x) = x^\tau$ for $\tau < 0$, and contradicts to the above dense graph condition otherwise.

Therefore, $f(x) = x^\tau$ for $x > 0$ and some constant $\tau > 0$. Dividing $E(a, b, c)$ by $(a - b)(b - c)(c - a) = (ab^2 + bc^2 + ca^2) - (a^2b + b^2c + c^2a)$ and taking a limit when a, b, c all go to 1 (the divided ratios tend to the corresponding derivatives, say, $\frac{a^\tau - b^\tau}{a-b} \to (x^\tau)'_{x=1} = \tau$), we get $\tau^3 = \tau \cdot 3^{\tau-1}$, $\tau^2 = 3^{\tau-1}$, $F(\tau) := 3^{\tau/2-1/2} - \tau = 0$. Since function F is strictly convex, it has at most two roots, and we get $\tau \in \{1, 3\}$.

Combinatorics

C1. Let S be an infinite set of positive integers, such that there exist four pairwise distinct $a, b, c, d \in S$ with $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist three pairwise distinct $x, y, z \in S$ such that $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.

Solution. There exists $\alpha \in S$ so that $\{\gcd(\alpha, s) \mid s \in S,\ s \neq \alpha\}$ contains at least two elements. Since α has only finitely many divisors, there is a $d \mid \alpha$ such that the set $B = \{\beta \in S \mid \gcd(\alpha, \beta) = d\}$ is infinite. Pick $\gamma \in S$ so that $\gcd(\alpha, \gamma) \neq d$. Pick $\beta_1, \beta_2 \in B$ so that $\gcd(\beta_1, \gamma) = \gcd(\beta_2, \gamma) =: d'$. If $d = d'$, then $\gcd(\alpha, \beta_1) = \gcd(\gamma, \beta_1) \neq \gcd(\alpha, \gamma)$. If $d \neq d'$, then either $\gcd(\alpha, \beta_1) = \gcd(\alpha, \beta_2) = d$ and $\gcd(\beta_1, \beta_2) \neq d$ or $\gcd(\gamma, \beta_1) = \gcd(\gamma, \beta_2) = d'$ and $\gcd(\beta_1, \beta_2) \neq d'$.

Comment. The situation can be modelled as a complete graph on the infinite vertex set S, where every edge $\{s, t\}$ is colored by $c(s, t) := \gcd(s, t)$. For every vertex the incident edges carry only finitely many different colors, and by the problem statement at least two different colors show up on the edge set. The goal is to show that there exists a bi-colored triangle (a triangle, whose edges carry exactly two different colors).

For the proof, consider a vertex v whose incident edges carry at least two different colors. Let $X \subset S$ be an infinite subset so that $c(v, x) \equiv c_1$ for all $x \in X$. Let $y \in S$ be a vertex so that $c(v, y) \neq c_1$. Let $x_1, x_2 \in X$ be two vertices with $c(y, x_1) = c(y, x_2) = c_2$. If $c_1 = c_2$, then the triangle v, y, x_1 is bi-colored. If $c_1 \neq c_2$, then one of v, x_1, x_2 and y, x_1, x_2 is bi-colored.

C2. Let $n \geqslant 3$ be an integer. An integer $m \geqslant n+1$ is called *n-colourful* if, given infinitely many marbles in each of n colours $C_1, C_2, \ldots, C_n$, it is possible to place m of them around a circle so that in any group of $n+1$ consecutive marbles there is at least one marble of colour C_i for each $i = 1, \ldots, n$.

Prove that there are only finitely many positive integers which are not n-colourful. Find the largest among them.

Answer: $m_{max} = n^2 - n - 1$.

Solution. First suppose that there are $n(n-1)-1$ marbles. Then for one of the colours, say blue, there are at most $n-2$ marbles, which partition the non-blue marbles into at most $n-2$ groups with at least $(n-1)^2 > n(n-2)$ marbles in total. Thus one of these groups contains at least $n+1$ marbles and this group does not contain any blue marble.

Now suppose that the total number of marbles is at least $n(n-1)$. Then we may write this total number as $nk+j$ with some $k \geqslant n-1$ and with $0 \leqslant j \leqslant n-1$. We place around a circle $k-j$ copies of the colour sequence $[1, 2, 3, \ldots, n]$ followed by j copies of the colour sequence $[1, 1, 2, 3, \ldots, n]$.

C3. A thimblerigger has 2021 thimbles numbered from 1 through 2021. The thimbles are arranged in a circle in arbitrary order. The thimblerigger performs a sequence of 2021 moves; in the k^{th} move, he swaps the positions of the two thimbles adjacent to thimble k.

Prove that there exists a value of k such that, in the k^{th} move, the thimblerigger swaps some thimbles a and b such that $a < k < b$.

Solution. Assume the contrary. Say that the k^{th} thimble is the *central thimble* of the k^{th} move, and its position on that move is the *central position* of the move.

Step 1: Black and white colouring.

Before the moves start, let us paint all thimbles in white. Then, after each move, we repaint its central thimble in black. This way, at the end of the process all thimbles have become black.

By our assumption, in every move k, the two swapped thimbles have the same colour (as their numbers are either both larger or both smaller than k). At every moment, assign the colours of the thimbles to their current positions; then the only position which changes its colour in a move is its central position. In particular, each position is central for exactly one move (when it is being repainted to black).

Step 2: Red and green colouring.

Now we introduce a colouring of the *positions*. If in the k^{th} move, the numbers of the two swapped thimbles are both less than k, then we paint the central position of the move in red; otherwise we paint that position in green. This way, each position has been painted in red or green exactly once. We claim that among any two adjacent positions, one becomes green and the other one becomes red; this will provide the desired contradiction since 2021 is odd.

Consider two adjacent positions A and B, which are central in the a^{th} and in the b^{th} moves, respectively, with $a < b$. Then in the a^{th} move the thimble at position B is white, and therefore has a number greater than a. After the a^{th} move, position A is green and the thimble at position A is black. By the arguments from Step 1, position A contains only black thimbles after the a^{th} step. Therefore, on the b^{th} move, position A contains a black thimble whose number is therefore less than b, while thimble b is at position B. So position B becomes red, and hence A and B have different colours.

Comment 1. Essentially, Step 1 provides the proof of the following two assertions (under the indirect assumption):

(1) Each position P becomes central in exactly one move (denote that move's number by k); and

(2) Before the k^{th} move, position P always contains a thimble whose number is larger than the number of the current move, while after the k^{th} move the position always contains a thimble whose number is smaller than the number of the current move.

Both (1) and (2) can be proved without introduction of colours, yet the colours help to visualise the argument.

After these two assertions have been proved, Step 2 can be performed in various ways, e.g., as follows.

At any moment in the process, the black positions are split into several groups consisting of one or more contiguous black positions each; different groups are separated by white positions. Now one can prove by induction on k that, after the k^{th} move, all groups have odd sizes. Indeed, in every move, the new black position either forms a separate group, or merges two groups (say, of lengths a and b) into a single group of length $a + b + 1$.

However, after the 2020^{th} move the black positions should form one group of length 2020. This is a contradiction.

This argument has several variations; e.g., one can check in a similar way that, after the process starts, at least one among the groups of *white* positions has an even size.

Comment 2. The solution above works equally well for any *odd* number of thimbles greater than 1, instead of 2021. On the other hand, a similar statement with an *even* number $n = 2k \geqslant 4$ of thimbles is wrong. To show that, the thimblerigger can enumerate positions from 1 through n clockwise, and then put thimbles $1, 2, \ldots, k$ at the odd positions, and thimbles $k+1, k+2, \ldots, 2k$ at the even positions.

C4. The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a *path from X to Y* is a sequence of roads such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called *diverse* if no road belongs to two or more paths in the collection.

Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

Solution 1. We write $X \to Y$ (or $Y \leftarrow X$) if the road between X and Y goes from X to Y. Notice that, if there is any route moving from X to Y (possibly passing through some cities more than once), then there is a path from X to Y consisting of some roads in the route. Indeed, any cycle in the route may be removed harmlessly; after some removals one obtains a path.

Say that a path is *short* if it consists of one or two roads.

Partition all cities different from A and B into four groups, $\mathcal{I}$, $\mathcal{O}$, $\mathcal{A}$, and $\mathcal{B}$ according to the following rules: for each city C,

$$C \in \mathcal{I} \iff A \to C \leftarrow B; \qquad\qquad C \in \mathcal{O} \iff A \leftarrow C \to B;$$
$$C \in \mathcal{A} \iff A \to C \to B; \qquad\qquad C \in \mathcal{B} \iff A \leftarrow C \leftarrow B.$$

Lemma. Let $\mathcal{P}$ be a diverse collection consisting of p paths from A to B. Then there exists a diverse collection consisting of at least p paths from A to B and containing all short paths from A to B.

Proof. In order to obtain the desired collection, modify $\mathcal{P}$ as follows.

If there is a direct road $A \to B$ and the path consisting of this single road is not in $\mathcal{P}$, merely add it to $\mathcal{P}$.

Now consider any city $C \in \mathcal{A}$ such that the path $A \to C \to B$ is not in $\mathcal{P}$. If $\mathcal{P}$ contains at most one path containing a road $A \to C$ or $C \to B$, remove that path (if it exists), and add the path $A \to C \to B$ to $\mathcal{P}$ instead. Otherwise, $\mathcal{P}$ contains two paths of the forms $A \to C \dashrightarrow B$ and $A \dashrightarrow C \to B$, where $C \dashrightarrow B$ and $A \dashrightarrow C$ are some paths. In this case, we recombine the edges to form two new paths $A \to C \to B$ and $A \dashrightarrow C \dashrightarrow B$ (removing cycles from the latter if needed). Now we replace the old two paths in $\mathcal{P}$ with the two new ones.

After any operation described above, the number of paths in the collection does not decrease, and the collection remains diverse. Applying such operation to each $C \in \mathcal{A}$, we obtain the desired collection. □

Back to the problem, assume, without loss of generality, that there is a road $A \to B$, and let a and b denote the numbers of roads going out from A and B, respectively. Choose a diverse collection $\mathcal{P}$ consisting of N_{AB} paths from A to B. We will transform it into a diverse collection $\mathcal{Q}$ consisting of at least $N_{AB} + (b - a)$ paths from B to A. This construction yields

$$N_{BA} \geqslant N_{AB} + (b - a); \qquad \text{similarly, we get} \quad N_{AB} \geqslant N_{BA} + (a - b),$$

whence $N_{BA} - N_{AB} = b - a$. This yields the desired equivalence.

Apply the lemma to get a diverse collection $\mathcal{P}'$ of at least N_{AB} paths containing all $|\mathcal{A}| + 1$ short paths from A to B. Notice that the paths in $\mathcal{P}'$ contain no edge of a short path from B to A. Each non-short path in $\mathcal{P}'$ has the form $A \to C \dashrightarrow D \to B$, where $C \dashrightarrow D$ is a path from some city $C \in \mathcal{I}$ to some city $D \in \mathcal{O}$. For each such path, put into $\mathcal{Q}$ the path $B \to C \dashrightarrow D \to A$; also put into $\mathcal{Q}$ all short paths from B to A. Clearly, the collection $\mathcal{Q}$ is diverse.

Now, all roads going out from A end in the cities from $\mathcal{I} \cup \mathcal{A} \cup \{B\}$, while all roads going out from B end in the cities from $\mathcal{I} \cup \mathcal{B}$. Therefore,

$$a = |\mathcal{I}| + |\mathcal{A}| + 1, \quad b = |\mathcal{I}| + |\mathcal{B}|, \qquad \text{and hence} \quad a - b = |\mathcal{A}| - |\mathcal{B}| + 1.$$

On the other hand, since there are $|\mathcal{A}| + 1$ short paths from A to B (including $A \to B$) and $|\mathcal{B}|$ short paths from B to A, we infer

$$|\mathcal{Q}| = |\mathcal{P}'| - (|\mathcal{A}| + 1) + |\mathcal{B}| \geqslant N_{AB} + (b - a),$$

as desired.

Solution 2. We recall some graph-theoretical notions. Let G be a finite graph, and let V be the set of its vertices; fix two distinct vertices $s, t \in V$. An (s,t)-*cut* is a partition of V into two parts $V = S \sqcup T$ such that $s \in S$ and $t \in T$. The *cut-edges* in the cut (S,T) are the edges going from S to T, and the *size* $e(S,T)$ of the cut is the number of cut-edges.

We will make use of the following theorem (which is a partial case of the Ford–Fulkerson "min-cut max-flow" theorem).

Theorem (Menger). Let G be a directed graph, and let s and t be its distinct vertices. Then the maximal number of edge-disjoint paths from s to t is equal to the minimal size of an (s,t)-cut.

Back to the problem. Consider a directed graph G whose vertices are the cities, and edges correspond to the roads. Then N_{AB} is the maximal number of edge-disjoint paths from A to B in this graph; the number N_{BA} is interpreted similarly.

As in the previous solution, denote by a and b the out-degrees of vertices A and B, respectively. To solve the problem, we show that for any (A,B)-cut (S_A, T_A) in our graph there exists a (B,A)-cut (S_B, T_B) satisfying

$$e(S_B, T_B) = e(S_A, T_A) + (b - a).$$

This yields

$$N_{BA} \leqslant N_{AB} + (b - a); \qquad \text{similarly, we get} \quad N_{AB} \leqslant N_{BA} + (a - b),$$

whence again $N_{BA} - N_{AB} = b - a$.

The construction is simple: we put $S_B = S_A \cup \{B\} \setminus \{A\}$ and hence $T_B = T_A \cup \{A\} \setminus \{B\}$. To show that it works, let A and B denote the sets of cut-edges in (S_A, T_A) and (S_B, T_B), respectively. Let a_s and $a_t = a - a_s$ denote the numbers of edges going from A to S_A and T_A, respectively. Similarly, denote by b_s and $b_t = b - b_s$ the numbers of edges going from B to S_B and T_B, respectively.

Notice that any edge incident to none of A and B either belongs to both A and B, or belongs to none of them. Denote the number of such edges belonging to A by c. The edges in A which are not yet accounted for split into two categories: those going out from A to T_A (including $A \to B$ if it exists), and those going from $S_A \setminus \{A\}$ to B — in other words, going from S_B to B. The numbers of edges in the two categories are a_t and $|S_B| - 1 - b_s$, respectively. Therefore,

$$|\mathsf{A}| = c + a_t + (|S_B| - b_s - 1). \qquad \text{Similarly, we get} \quad |\mathsf{B}| = c + b_t + (|S_A| - a_s - 1),$$

and hence

$$|\mathsf{B}| - |\mathsf{A}| = (b_t + b_s) - (a_t + a_s) = b - a,$$

since $|S_A| = |S_B|$. This finishes the solution.

C5. Let n and k be two integers with $n > k \geqslant 1$. There are $2n + 1$ students standing in a circle. Each student S has $2k$ *neighbours*— namely, the k students closest to S on the right, and the k students closest to S on the left.

Suppose that $n + 1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbours.

Solution. We replace the girls by 1's, and the boys by 0's, getting the numbers $a_1, a_2, \ldots, a_{2n+1}$ arranged in a circle. We extend this sequence periodically by letting $a_{2n+1+k} = a_k$ for all $k \in \mathbb{Z}$. We get an infinite periodic sequence

$$\ldots, a_1, a_2, \ldots, a_{2n+1}, a_1, a_2, \ldots, a_{2n+1}, \ldots.$$

Consider the numbers $b_i = a_i + a_{i-k-1} - 1 \in \{-1, 0, 1\}$ for all $i \in \mathbb{Z}$. We know that

$$b_{m+1} + b_{m+2} + \cdots + b_{m+2n+1} = 1 \qquad (m \in \mathbb{Z}); \tag{1}$$

in particular, this yields that there exists some i_0 with $b_{i_0} = 1$. Now we want to find an index i such that

$$b_i = 1 \quad \text{and} \quad b_{i+1} + b_{i+2} + \cdots + b_{i+k} \geqslant 0. \tag{2}$$

This will imply that $a_i = 1$ and

$$(a_{i-k} + a_{i-k+1} + \cdots + a_{i-1}) + (a_{i+1} + a_{i+2} + \cdots + a_{i+k}) \geqslant k,$$

as desired.

Suppose, to the contrary, that for every index i with $b_i = 1$ the sum $b_{i+1} + b_{i+2} + \cdots + b_{i+k}$ is negative. We start from some index i_0 with $b_{i_0} = 1$ and construct a sequence i_0, i_1, i_2, $\ldots$, where i_j $(j > 0)$ is the smallest possible index such that $i_j > i_{j-1} + k$ and $b_{i_j} = 1$. We can choose two numbers among i_0, i_1, $\ldots$, i_{2n+1} which are congruent modulo $2n + 1$ (without loss of generality, we may assume that these numbers are i_0 and i_T).

On the one hand, for every j with $0 \leqslant j \leqslant T - 1$ we have

$$S_j := b_{i_j} + b_{i_j+1} + b_{i_j+2} + \cdots + b_{i_{j+1}-1} \leqslant b_{i_j} + b_{i_j+1} + b_{i_j+2} + \cdots + b_{i_j+k} \leqslant 0$$

since $b_{i_j+k+1}, \ldots, b_{i_{j+1}-1} \leqslant 0$. On the other hand, since $(i_T - i_0) \mid (2n + 1)$, from (1) we deduce

$$S_0 + \cdots + S_{T-1} = \sum_{i=i_0}^{i_T-1} b_i = \frac{i_T - i_0}{2n + 1} > 0.$$

This contradiction finishes the solution.

Comment 1. After the problem is reduced to finding an index i satisfying (2), one can finish the solution by applying the (existence part of) following statement.

Lemma (Raney). If $\langle x_1, x_2, \ldots, x_m \rangle$ is any sequence of integers whose sum is $+1$, exactly one of the cyclic shifts $\langle x_1, x_2, \ldots, x_m \rangle, \langle x_2, \ldots, x_m, x_1 \rangle, \ldots, \langle x_m, x_1, \ldots, x_{m-1} \rangle$ has all of its partial sums positive.

A (possibly wider known) version of this lemma, which also can be used in order to solve the problem, is the following

Claim (Gas stations problem). Assume that there are several fuel stations located on a circular route which together contain just enough gas to make one trip around. Then one can make it all the way around, starting at the right station with an empty tank.

Both Raney's theorem and the Gas stations problem admit many different (parallel) proofs. Their ideas can be disguised in direct solutions of the problem at hand (as it, in fact, happens in the above solution); such solutions may avoid the introduction of the b_i. Below, in Comment 2 we present a variant of such solution, while in Comment 3 we present an alternative proof of Raney's theorem.

Comment 2. Here is a version of the solution which avoids the use of the b_i.

Suppose the contrary. Introduce the numbers a_i as above. Starting from any index s_0 with $a_{s_0} = 1$, we construct a sequence $s_0, s_1, s_2, \ldots$ by letting s_i to be the smallest index larger than $s_{i-1}+k$ such that $a_{s_i} = 1$, for $i = 1, 2, \ldots$. Choose two indices among $s_1, \ldots, s_{2n+1}$ which are congruent modulo $2n+1$; we assume those two are s_0 and s_T, with $s_T - s_0 = t(2n+1)$. Notice here that $s_{T+1} - s_T = s_1 - s_0$.

For every $i = 0, 1, 2, \ldots, T$, put

$$L_i = s_{i+1} - s_i \quad \text{and} \quad S_i = a_{s_i} + a_{s_i+1} + \cdots + a_{s_{i+1}-1}.$$

Now, by the indirect assumption, for every $i = 1, 2, \ldots, T$, we have

$$a_{s_i-k} + a_{s_i-k+1} + \cdots + a_{s_i+k} \leqslant a_{s_i} + (k-1) = k.$$

Recall that $a_j = 0$ for all j with $s_i + k < j < a_{s_{i+1}}$. Therefore,

$$S_{i-1} + S_i = \sum_{j=s_{i-1}}^{s_i+k} a_j = \sum_{j=s_{i-1}}^{s_i-k-1} a_j + \sum_{j=s_i-k}^{s_i+k} a_j \leqslant (s_i - s_{i-1} - k) + k = L_{i-1}.$$

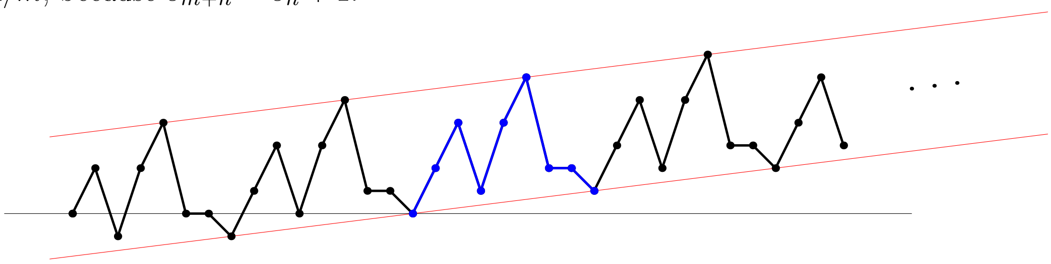
Summing up these equalities over $i = 1, 2, \ldots, T$ we get

$$2t(n+1) = \sum_{i=1}^{T}(S_{i-1} + S_i) \leqslant \sum_{i=1}^{T} L_{i-1} = (2n+1)t,$$

which is a contradiction.

Comment 3. Here we present a proof of Raney's lemma different from the one used above.

If we plot the partial sums $s_n = x_1 + \cdots + x_n$ as a function of n, the graph of s_n has an average slope of $1/m$, because $s_{m+n} = s_n + 1$.



The entire graph can be contained between two lines of slope $1/m$. In general these bounding lines touch the graph just once in each cycle of m points, since lines of slope $1/m$ hit points with integer coordinates only once per m units. The unique (in one cycle) lower point of intersection is the only place in the cycle from which all partial sums will be positive.

Comment 4. The following example shows that for different values of k the required girl may have different positions: 0110 0110 1.

C6. A hunter and an invisible rabbit play a game on an infinite square grid. First the hunter fixes a colouring of the cells with finitely many colours. The rabbit then secretly chooses a cell to start in. Every minute, the rabbit reports the colour of its current cell to the hunter, and then secretly moves to an adjacent cell that it has not visited before (two cells are adjacent if they share a side). The hunter wins if after some finite time either

- the rabbit cannot move; or
- the hunter can determine the cell in which the rabbit started.

Decide whether there exists a winning strategy for the hunter.

Answer: Yes, there exists a colouring that yields a winning strategy for the hunter.

Solution. A central idea is that several colourings $C_1, C_2, \ldots, C_k$ can be merged together into a single product colouring $C_1 \times C_2 \times \cdots \times C_k$ as follows: the colours in the product colouring are ordered tuples $(c_1, \ldots, c_n)$ of colours, where c_i is a colour used in C_i, so that each cell gets a tuple consisting of its colours in the individual colourings C_i. This way, any information which can be determined from one of the individual colourings can also be determined from the product colouring.

Now let the hunter merge the following colourings:

- The first two colourings C_1 and C_2 allow the tracking of the horizontal and vertical movements of the rabbit.

  The colouring C_1 colours the cells according to the residue of their x-coordinates modulo 3, which allows to determine whether the rabbit moves left, moves right, or moves vertically. Similarly, the colouring C_2 uses the residues of the y-coordinates modulo 3, which allows to determine whether the rabbit moves up, moves down, or moves horizontally.

- Under the condition that the rabbit's x-coordinate is unbounded, colouring C_3 allows to determine the exact value of the x-coordinate:

  In C_3, the columns are coloured white and black so that the gaps between neighboring black columns are pairwise distinct. As the rabbit's x-coordinate is unbounded, it will eventually visit two black cells in distinct columns. With the help of colouring C_1 the hunter can catch that moment, and determine the difference of x-coordinates of those two black cells, hence deducing the precise column.

  Symmetrically, under the condition that the rabbit's y-coordinate is unbounded, there is a colouring C_4 that allows the hunter to determine the exact value of the y-coordinate.

- Finally, under the condition that the sum $x + y$ of the rabbit's coordinates is unbounded, colouring C_5 allows to determine the exact value of this sum: The diagonal lines $x + y =$ const are coloured black and white, so that the gaps between neighboring black diagonals are pairwise distinct.

Unless the rabbit gets stuck, at least two of the three values x, y and $x+y$ must be unbounded as the rabbit keeps moving. Hence the hunter can eventually determine two of these three values; thus he does know all three. Finally the hunter works backwards with help of the colourings C_1 and C_2 and computes the starting cell of the rabbit.

Comment. There are some variations of the solution above: e.g., the colourings C_3, C_4 and C_5 can be replaced with different ones. However, such alternatives are more technically involved, and we do not present them here.

C7. Consider a checkered $3m \times 3m$ square, where m is an integer greater than 1. A frog sits on the lower left corner cell S and wants to get to the upper right corner cell F. The frog can hop from any cell to either the next cell to the right or the next cell upwards.

Some cells can be sticky, and the frog gets trapped once it hops on such a cell. A set X of cells is called *blocking* if the frog cannot reach F from S when all the cells of X are sticky. A blocking set is *minimal* if it does not contain a smaller blocking set.

(a) Prove that there exists a minimal blocking set containing at least $3m^2 - 3m$ cells.

(b) Prove that every minimal blocking set contains at most $3m^2$ cells.

Note. An example of a minimal blocking set for $m = 2$ is shown below. Cells of the set X are marked by letters x.

					F
x	x				
		x			
			x		
				x	
S		x			

Solution for part (a). In the following example the square is divided into m stripes of size $3 \times 3m$. It is easy to see that X is a minimal blocking set. The first and the last stripe each contains $3m - 1$ cells from the set X; every other stripe contains $3m - 2$ cells, see Figure 1. The total number of cells in the set X is $3m^2 - 2m + 2$.

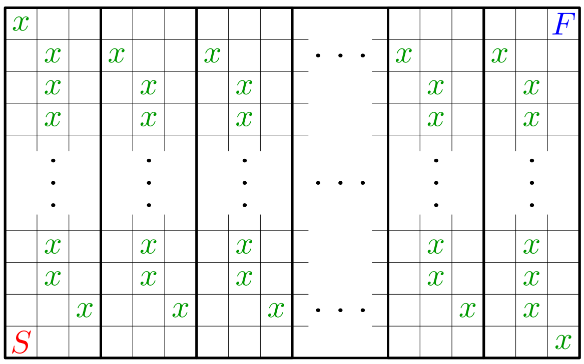


Figure 1

Solution 1 for part (b). For a given blocking set X, say that a non-sticky cell is *red* if the frog can reach it from S via some hops without entering set X. We call a non-sticky cell *blue* if the frog can reach F from that cell via hops without entering set X. One can regard the blue cells as those reachable from F by *anti-hops*, i.e. moves downwards and to the left. We also colour all cells in X *green*. It follows from the definition of the blocking set that no cell will be coloured twice. In Figure 2 we show a sample of a blocking set and the corresponding colouring.

Now assume that X is a minimal blocking set. We denote by R (resp., B and G) be the total number of red (resp., blue and green) cells.

We claim that $G \leqslant R + 1$ and $G \leqslant B + 1$. Indeed, there are at most $2R$ possible frog hops from red cells. Every green or red cell (except for S) is accessible by such hops. Hence $2R \geqslant G + (R - 1)$, or equivalently $G \leqslant R + 1$. In order to prove the inequality $G \leqslant B + 1$, we turn over the board and apply the similar arguments.

Therefore we get $9m^2 \geqslant B + R + G \geqslant 3G - 2$, so $G \leqslant 3m^2$.

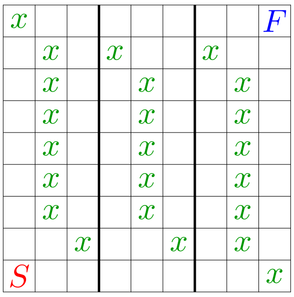


Figure 2 (a)

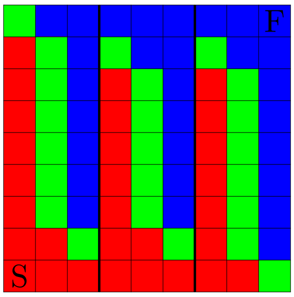
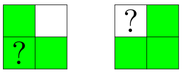


Figure 2 (b)

Solution 2 for part (b). We shall use the same colouring as in the above solution. Again, assume that X is a minimal blocking set.

Note that any 2×2 square cannot contain more than 2 green cells. Indeed, on Figure 3(a) the cell marked with "?" does not block any path, while on Figure 3(b) the cell marked with "?" should be coloured red and blue simultaneously. So we can split all green cells into chains consisting of three types of links shown on Figure 4 (diagonal link in the other direction is not allowed, corresponding green cells must belong to different chains). For example, there are 3 chains in Figure 2(b).



(a) (b)

Figure 3

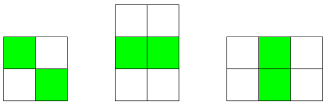


Figure 4

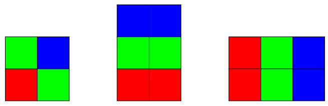
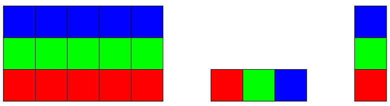


Figure 5

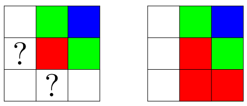
We will inscribe green chains in disjoint axis-aligned rectangles so that the number of green cells in each rectangle will not exceed 1/3 of the area of the rectangle. This will give us the bound $G \leqslant 3m^2$. Sometimes the rectangle will be the minimal bounding rectangle of the chain, sometimes minimal bounding rectangles will be expanded in one or two directions in order to have sufficiently large area.

Note that for any two consecutive cells in the chain the colouring of some neighbouring cells is uniquely defined (see Figure 5). In particular, this observation gives a corresponding rectangle for the chains of height (or width) 1 (see Figure 6(a)). A separate green cell can be inscribed in 1×3 or 3×1 rectangle with one red and one blue cell, see Figure 6(b)–(c), otherwise we get one of impossible configurations shown in Figure 3.



(a) (b) (c)

Figure 6



(a) (b)

Figure 7

Any diagonal chain of length 2 is always inscribed in a 2×3 or 3×2 rectangle without another green cells. Indeed, one of the squares marked with "?" in Figure 7(a) must be red. If it is the bottom question mark, then the remaining cell in the corresponding 2×3 rectangle must have the same colour, see Figure 7(b).

A longer chain of height (or width) 2 always has a horizontal (resp., vertical) link and can be inscribed into a $3 \times a$ rectangle. In this case we expand the minimal bounding rectangle across the long side which touches the mentioned link. On Figure 8(a) the corresponding expansion of the minimal bounding rectangle is coloured in light blue. The upper right corner cell must be also blue. Indeed it cannot be red or green. If it is not coloured in blue, see Figure 8(b), then all anti-hop paths from F to "?" are blocked with green cells. And these green cells are surrounded by blue ones, what is impossible. In this case the green chain contains a cells, which is exactly 1/3 of the area of the rectangle.

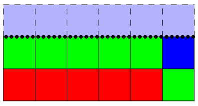


Figure 8 (a)

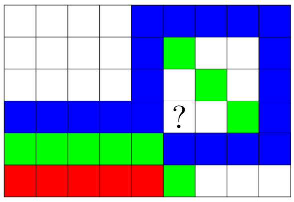


Figure 8 (b)

In the remaining case the minimal bounding rectangle of the chain is of size $a \times b$ where $a, b \geqslant 3$. Denote by ℓ the *length* of the chain (i.e. the number of cells in the chain).

If the chain has at least two diagonal links (see Figure 9), then $\ell \leqslant a + b - 3 \leqslant ab/3$.

If the chain has only one diagonal link then $\ell = a+b-2$. In this case the chain has horizontal and vertical end-links, and we expand the minimal bounding rectangle in two directions to get an $(a+1) \times (b+1)$ rectangle. On Figure 10 a corresponding expansion of the minimal bounding rectangle is coloured in light red. Again the length of the chain does not exceed $1/3$ of the rectangle's area: $\ell \leqslant a + b - 2 \leqslant (a+1)(b+1)/3$.

On the next step we will use the following statement: all cells in constructed rectangles are coloured red, green or blue (the cells upwards and to the right of green cells are blue; the cells downwards and to the left of green cells are red). The proof repeats the same arguments as before (see Figure 8(b).)

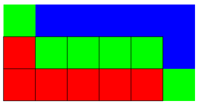


Figure 9

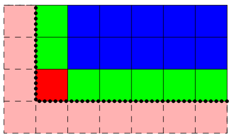
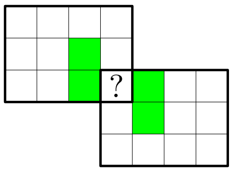
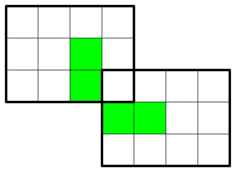


Figure 10



(a)



(b)

Figure 11

Note that all constructed rectangles are disjoint. Indeed, assume that two rectangles have a common cell. Using the above statement, one can see that the only such cell can be a common corner cell, as shown in Figure 11. Moreover, in this case both rectangles should be expanded, otherwise they would share a green corner cell.

If they were expanded along the same axis (see Figure 11(a)), then again the common corner cannot be coloured correctly. If they were expanded along different axes (see Figure 11(b)) then the two chains have a common point and must be connected in one chain. (These arguments work for 2×3 and 1×3 rectangles in a similar manner.)

Comment 1. We do not *a priori* know whether all points are either red, or blue, or green. One might colour the remaining cells in black. The arguments from Solution 2 allow to prove that black cells do not exist. (One can start with a black cell which is nearest to S. Its left and downward neighbours must be coloured green or blue. In all cases one gets a configuration similar to Figure 8(b).)

Comment 2. The maximal possible size of a minimal blocking set in $3m \times 3m$ rectangle seems to be $3m^2 - 2m + 2$.

One can prove a more precise upper bound on the cardinality of the minimal blocking set: $G \leqslant 3m^2 - m + 2$. Denote by D_R the number of red branching cells (i.e. such cells which have 2 red subsequent neighbours). And let D_B be the number of similar blue cells. Then a double counting argument allows to prove that $G \leqslant R - D_R + 1$ and $G \leqslant B - D_B + 1$. Thus, we can bound G in terms of D_B and D_R as

$$9m^2 \geqslant R + B + G \geqslant 3G + D_R + D_B - 2.$$

Now one can estimate the number of branching cells in order to obtain that $G \leqslant 3m^2 - m + 2$.

Comment 3. An example with $3m^2 - 2m + 2$ green cells may look differently; see, e.g., Figure 12.

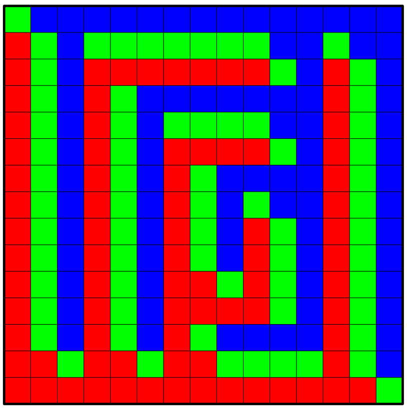


Figure 12

C8. Determine the largest N for which there exists a table T of integers with N rows and 100 columns that has the following properties:

(i) Every row contains the numbers 1, 2, ..., 100 in some order.

(ii) For any two distinct rows r and s, there is a column c such that $|T(r,c) - T(s,c)| \geqslant 2$.

Here $T(r,c)$ means the number at the intersection of the row r and the column c.

Answer: The largest such integer is $N = 100!/2^{50}$.

Solution 1.

Non-existence of a larger table. Let us consider some fixed row in the table, and let us replace (for $k = 1, 2, \ldots, 50$) each of two numbers $2k-1$ and $2k$ respectively by the symbol x_k. The resulting *pattern* is an arrangement of 50 symbols $x_1, x_2, \ldots, x_{50}$, where every symbol occurs exactly twice. Note that there are $N = 100!/2^{50}$ distinct patterns $P_1, \ldots, P_N$.

If two rows $r \neq s$ in the table have the same pattern P_i, then $|T(r,c) - T(s,c)| \leqslant 1$ holds for all columns c. As this violates property (ii) in the problem statement, different rows have different patterns. Hence there are at most $N = 100!/2^{50}$ rows.

Existence of a table with N rows. We construct the table by translating every pattern P_i into a corresponding row with the numbers $1, 2, \ldots, 100$. We present a procedure that inductively replaces the symbols by numbers. The translation goes through steps $k = 1, 2, \ldots, 50$ in increasing order and at step k replaces the two occurrences of symbol x_k by $2k-1$ and $2k$.

- The left occurrence of x_1 is replaced by 1, and its right occurrence is replaced by 2.
- For $k \geqslant 2$, we already have the number $2k-2$ somewhere in the row, and now we are looking for the places for $2k-1$ and $2k$. We make the three numbers $2k-2, 2k-1, 2k$ show up (ordered from left to right) either in the order $2k-2, 2k-1, 2k$, or as $2k, 2k-2, 2k-1$, or as $2k-1, 2k, 2k-2$. This is possible, since the number $2k-2$ has been placed in the preceding step, and shows up before / between / after the two occurrences of the symbol x_k.

We claim that the N rows that result from the N patterns yield a table with the desired property (ii). Indeed, consider the r-th and the s-th row ($r \neq s$), which by construction result from patterns P_r and P_s. Call a symbol x_i *aligned*, if it occurs in the same two columns in P_r and in P_s. Let k be the largest index, for which symbol x_k is not aligned. Note that $k \geqslant 2$. Consider the column c' with $T(r,c') = 2k$ and the column c'' with $T(s,c'') = 2k$. Then $T(r,c'') \leqslant 2k$ and $T(s,c') \leqslant 2k$, as all symbols x_i with $i \geqslant k+1$ are aligned.

- If $T(r,c'') \leqslant 2k-2$, then $|T(r,c'') - T(s,c'')| \geqslant 2$ as desired.
- If $T(s,c') \leqslant 2k-2$, then $|T(r,c') - T(s,c')| \geqslant 2$ as desired.
- If $T(r,c'') = 2k-1$ and $T(s,c') = 2k-1$, then the symbol x_k is aligned; contradiction.

In the only remaining case we have $c' = c''$, so that $T(r,c') = T(s,c') = 2k$ holds. Now let us consider the columns d' and d'' with $T(r,d') = 2k-1$ and $T(s,d'') = 2k-1$. Then $d \neq d''$ (as the symbol x_k is not aligned), and $T(r,d'') \leqslant 2k-2$ and $T(s,d') \leqslant 2k-2$ (as all symbols x_i with $i \geqslant k+1$ are aligned).

- If $T(r,d'') \leqslant 2k-3$, then $|T(r,d'') - T(s,d'')| \geqslant 2$ as desired.
- If $T(s,c') \leqslant 2k-3$, then $|T(r,d') - T(s,d')| \geqslant 2$ as desired.

In the only remaining case we have $T(r,d'') = 2k-2$ and $T(s,d') = 2k-2$. Now the row r has the numbers $2k-2, 2k-1, 2k$ in the three columns d', d'', c'. As one of these triples violates the ordering property of $2k-2, 2k-1, 2k$, we have the final contradiction. □

Comment 1. We can identify rows of the table T with permutations of $\mathcal{M} := \{1, \dots, 100\}$; also for every set $S \subset \mathcal{M}$ each row induces a subpermutation of S obtained by ignoring all entries not from S.

The example from Solution 1 consists of all permutations for which all subpermutations of the 50 sets $\{1,2\}, \{2,3,4\}, \{4,5,6\}, \dots, \{98,99,100\}$ are even.

Solution 2. We provide a bit different proof why the example from Solution 1 (see also Comment 1) works.

Lemma. Let π_1 and π_2 be two permutations of the set $\{1, 2, \dots, n\}$ such that $|\pi_1(i) - \pi_2(i)| \leqslant 1$ for every i. Then there exists a set of disjoint pairs $(i, i+1)$ such that π_2 is obtained from π_1 by swapping elements in each pair from the set.

Proof. We may assume that $\pi_1(i) = i$ for every i and proceed by induction on n. The case $n = 1$ is trivial. If $\pi_2(n) = n$, we simply apply the induction hypothesis. If $\pi_2(n) = n-1$, then $\pi_2(i) = n$ for some $i < n$. It is clear that $i = n-1$, and we can also use the induction hypothesis. $\square$

Now let π_1 and π_2 be two rows (which we identify with permutations of $\{1, 2, \dots, 100\}$) of the table constructed in Solution 1. Assume that $|\pi_1(i) - \pi_2(i)| \leqslant 1$ for any i. From the Lemma it follows that there exists a set $S \subset \{1, \dots, 99\}$ such that any two numbers from S differ by at least 2 and π_2 is obtained from π_1 by applying the permutations $(j, j+1)$, $j \in S$. Let $r = \min(S)$. If $r = 2k-1$ is odd, then π_1 and π_2 induce two subpermutations of $\{2k-2, 2k-1, 2k\}$ (or of $\{1,2\}$ for $k = 1$) of opposite parities. Thus $r = 2k$ is even. Since π_1 and π_2 induce subpermutations of the same (even) parity of $\{2k, 2k+1, 2k+2\}$, we must have $2k+2 \in S$. Next, $2k+4 \in S$ and so on, we get $98 \in S$, but then the parities of the subpermutations of $\{98, 99, 100\}$ in π_1, π_2 are opposite. A contradiction. $\square$

Comment 2. In Solution 2 we only used that for each set from $\{1,2\}, \{2,3,4\}, \{4,5,6\}, \dots, \{98,99,100\}$ any two rows of T induce a subpermutation of the same parity, not necessarily even.

This page is intentionally left blank

Geometry

G1. Let $ABCD$ be a parallelogram such that $AC = BC$. A point P is chosen on the extension of the segment AB beyond B. The circumcircle of the triangle ACD meets the segment PD again at Q, and the circumcircle of the triangle APQ meets the segment PC again at R. Prove that the lines CD, AQ, and BR are concurrent.

Common remarks. The introductory steps presented here are used in all solutions below.

Since $AC = BC = AD$, we have $\angle ABC = \angle BAC = \angle ACD = \angle ADC$. Since the quadrilaterals $APRQ$ and $AQCD$ are cyclic, we obtain

$$\angle CRA = 180^\circ - \angle ARP = 180^\circ - \angle AQP = \angle DQA = \angle DCA = \angle CBA,$$

so the points A, B, C, and R lie on some circle γ.

Solution 1. Introduce the point $X = AQ \cap CD$; we need to prove that B, R and X are collinear.

By means of the circle $(APRQ)$ we have

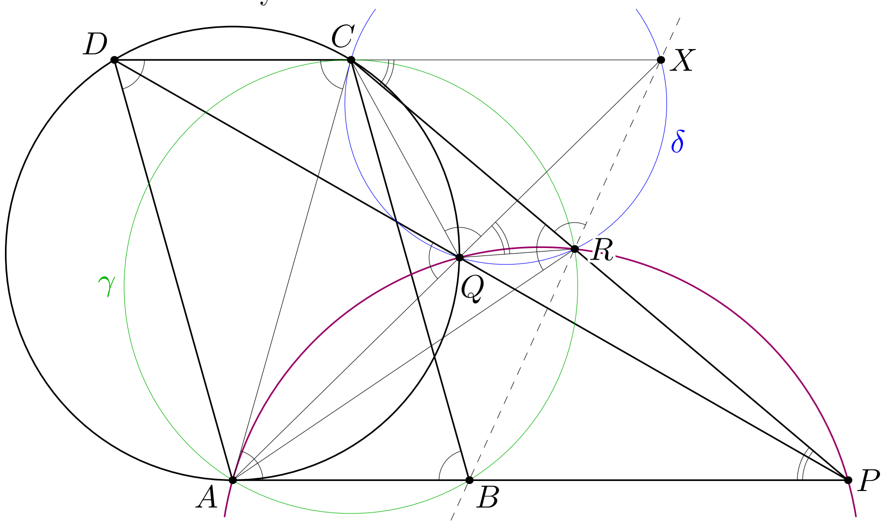
$$\angle RQX = 180^\circ - \angle AQR = \angle RPA = \angle RCX$$

(the last equality holds in view of $AB \parallel CD$), which means that the points C, Q, R, and X also lie on some circle δ.

Using the circles δ and γ we finally obtain

$$\angle XRC = \angle XQC = 180^\circ - \angle CQA = \angle ADC = \angle BAC = 180^\circ - \angle CRB,$$

that proves the desired collinearity.



Solution 2. Let α denote the circle $(APRQ)$. Since

$$\angle CAP = \angle ACD = \angle AQD = 180^\circ - \angle AQP,$$

the line AC is tangent to α.

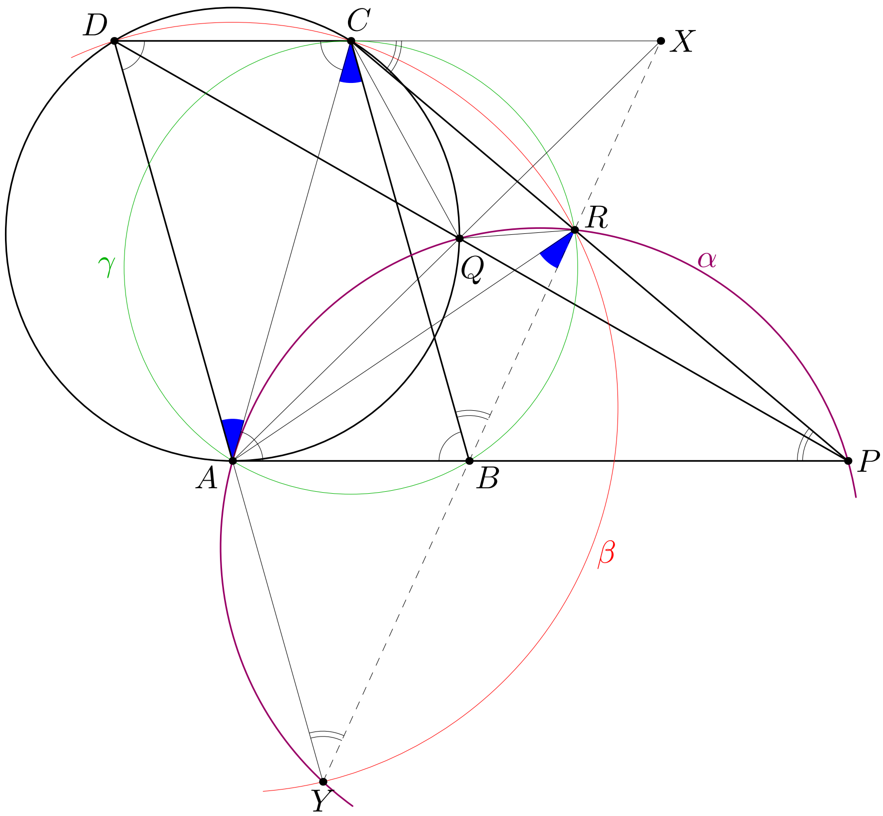
Now, let AD meet α again at a point Y (which necessarily lies on the extension of DA beyond A). Using the circle γ, along with the fact that AC is tangent to α, we have

$$\angle ARY = \angle CAD = \angle ACB = \angle ARB,$$

so the points Y, B, and R are collinear.

Applying Pascal's theorem to the hexagon $AAYRPQ$ (where AA is regarded as the tangent to α at A), we see that the points $AA \cap RP = C$, $AY \cap PQ = D$, and $YR \cap QA$ are collinear. Hence the lines CD, AQ, and BR are concurrent.

Comment 1. Solution 2 consists of two parts: (1) showing that BR and DA meet on α; and (2) showing that this yields the desired concurrency. Solution 3 also splits into those parts, but the proofs are different.



Solution 3. As in Solution 1, we introduce the point $X = AQ \cap CD$ and aim at proving that the points B, R, and X are collinear. As in Solution 2, we denote $\alpha = (APQR)$; but now we define Y to be the second meeting point of RB with α.

Using the circle α and noticing that CD is tangent to γ, we obtain

$$\angle RYA = \angle RPA = \angle RCX = \angle RBC. \tag{1}$$

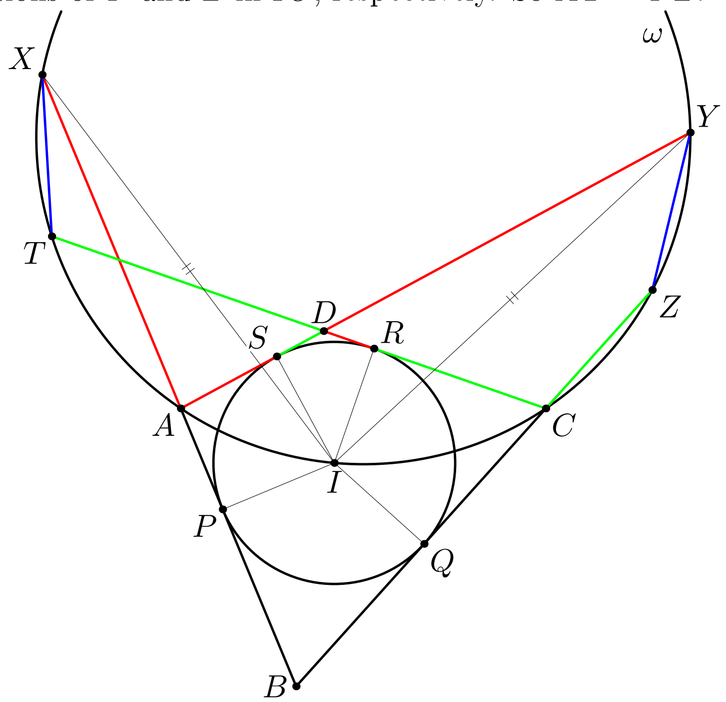
So $AY \parallel BC$, and hence Y lies on DA.

Now the chain of equalities (1) shows also that $\angle RYD = \angle RCX$, which implies that the points C, D, Y, and R lie on some circle β. Hence, the lines CD, AQ, and YBR are the pairwise radical axes of the circles $(AQCD)$, α, and β, so those lines are concurrent.

Comment 2. The original problem submission contained an additional assumption that $BP = AB$. The Problem Selection Committee removed this assumption as superfluous.

G2. Let $ABCD$ be a convex quadrilateral circumscribed around a circle with centre I. Let ω be the circumcircle of the triangle ACI. The extensions of BA and BC beyond A and C meet ω at X and Z, respectively. The extensions of AD and CD beyond D meet ω at Y and T, respectively. Prove that the perimeters of the (possibly self-intersecting) quadrilaterals $ADTX$ and $CDYZ$ are equal.

Solution. The point I is the intersection of the external bisector of the angle TCZ with the circumcircle ω of the triangle TCZ, so I is the midpoint of the arc TCZ and $IT = IZ$. Similarly, I is the midpoint of the arc YAX and $IX = IY$. Let O be the centre of ω. Then X and T are the reflections of Y and Z in IO, respectively. So $XT = YZ$.



Let the incircle of $ABCD$ touch AB, BC, CD, and DA at points P, Q, R, and S, respectively.

The right triangles IXP and IYS are congruent, since $IP = IS$ and $IX = IY$. Similarly, the right triangles IRT and IQZ are congruent. Therefore, $XP = YS$ and $RT = QZ$.

Denote the perimeters of $ADTX$ and $CDYZ$ by P_{ADTX} and P_{CDYZ} respectively. Since $AS = AP$, $CQ = RC$, and $SD = DR$, we obtain

$$\begin{aligned} P_{ADTX} &= XT + XA + AS + SD + DT = XT + XP + RT \\ &= YZ + YS + QZ = YZ + YD + DR + RC + CZ = P_{CDYZ}, \end{aligned}$$

as required.

Comment 1. After proving that X and T are the reflections of Y and Z in IO, respectively, one can finish the solution as follows. Since $XT = YZ$, the problem statement is equivalent to

$$XA + AD + DT = YD + DC + CZ. \tag{1}$$

Since $ABCD$ is circumscribed, $AB - AD = BC - CD$. Adding this to (1), we come to an equivalent equality $XA + AB + DT = YD + BC + CZ$, or

$$XB + DT = YD + BZ. \tag{2}$$

Let $\lambda = \frac{XZ}{AC} = \frac{TY}{AC}$. Since $XACZ$ is cyclic, the triangles ZBX and ABC are similar, hence

$$\frac{XB}{BC} = \frac{BZ}{AB} = \frac{XZ}{AC} = \lambda.$$

It follows that $XB = \lambda BC$ and $BZ = \lambda AB$. Likewise, the triangles TDY and ADC are similar, hence

$$\frac{DT}{AD} = \frac{DY}{CD} = \frac{TY}{AC} = \lambda.$$

Therefore, (2) rewrites as $\lambda BC + \lambda AD = \lambda CD + \lambda AB$.

This is equivalent to $BC + AD = CD + AB$ which is true as $ABCD$ is circumscribed.

Comment 2. Here is a more difficult modification of the original problem, found by the PSC.

Let $ABCD$ be a convex quadrilateral circumscribed around a circle with centre I. Let ω be the circumcircle of the triangle ACI. The extensions of BA and BC beyond A and C meet ω at X and Z, respectively. The extensions of AD and CD beyond D meet ω at Y and T, respectively. Let $U = BC \cap AD$ and $V = BA \cap CD$. Let I_U be the incentre of UYZ and let J_V be the V-excentre of VXT. Then $I_U J_V \perp BD$.

G3.

Version 1. Let n be a fixed positive integer, and let S be the set of points (x, y) on the Cartesian plane such that both coordinates x and y are nonnegative integers smaller than $2n$ (thus $|\mathsf{S}| = 4n^2$). Assume that $\mathcal{F}$ is a set consisting of n^2 quadrilaterals such that all their vertices lie in S, and each point in S is a vertex of exactly one of the quadrilaterals in $\mathcal{F}$.

Determine the largest possible sum of areas of all n^2 quadrilaterals in $\mathcal{F}$.

Version 2. Let n be a fixed positive integer, and let S be the set of points (x, y) on the Cartesian plane such that both coordinates x and y are nonnegative integers smaller than $2n$ (thus $|\mathsf{S}| = 4n^2$). Assume that $\mathcal{F}$ is a set of polygons such that all vertices of polygons in $\mathcal{F}$ lie in S, and each point in S is a vertex of exactly one of the polygons in $\mathcal{F}$.

Determine the largest possible sum of areas of all polygons in $\mathcal{F}$.

Answer for both Versions: The largest possible sum of areas is $\Sigma(n) := \frac{1}{3}n^2(2n+1)(2n-1)$.

Common remarks. Throughout all solutions, the area of a polygon P will be denoted by $[P]$.

We say that a polygon is *legal* if all its vertices belong to S. Let $O = \left(n - \frac{1}{2}, n - \frac{1}{2}\right)$ be the centre of S. We say that a legal square is *central* if its centre is situated at O. Finally, say that a set $\mathcal{F}$ of polygons is *acceptable* if it satisfies the problem requirements, i.e. if all polygons in $\mathcal{F}$ are legal, and each point in S is a vertex of exactly one polygon in $\mathcal{F}$. For an acceptable set $\mathcal{F}$, we denote by $\Sigma(\mathcal{F})$ the sum of areas of polygons in $\mathcal{F}$.

Solution 1, for both Versions. Each point in S is a vertex of a unique central square. Thus the set $\mathcal{G}$ of central squares is acceptable. We will show that

$$\Sigma(\mathcal{F}) \leqslant \Sigma(\mathcal{G}) = \Sigma(n), \tag{1}$$

thus establishing the answer.

We will use the following key lemma.

Lemma 1. Let $P = A_1A_2\ldots A_m$ be a polygon, and let O be an arbitrary point in the plane. Then

$$[P] \leqslant \frac{1}{2}\sum_{i=1}^{m} OA_i^2; \tag{2}$$

moreover, if P is a square centred at O, then the inequality (2) turns into an equality.

Proof. Put $A_{n+1} = A_1$. For each $i = 1, 2, \ldots, m$, we have

$$[OA_iA_{i+1}] \leqslant \frac{OA_i \cdot OA_{i+1}}{2} \leqslant \frac{OA_i^2 + OA_{i+1}^2}{4}.$$

Therefore,

$$[P] \leqslant \sum_{i=1}^{m}[OA_iA_{i+1}] \leqslant \frac{1}{4}\sum_{i=1}^{m}(OA_i^2 + OA_{i+1}^2) = \frac{1}{2}\sum_{i=1}^{m} OA_i^2,$$

which proves (2). Finally, all the above inequalities turn into equalities when P is a square centred at O. $\square$

Back to the problem, consider an arbitrary acceptable set $\mathcal{F}$. Applying Lemma 1 to each element in $\mathcal{F}$ and to each element in $\mathcal{G}$ (achieving equality in the latter case), we obtain

$$\Sigma(\mathcal{F}) \leqslant \frac{1}{2}\sum_{A \in \mathsf{S}} OA^2 = \Sigma(\mathcal{G}),$$

which establishes the left inequality in (1).

It remains to compute $\Sigma(\mathcal{G})$. We have

$$\begin{aligned}\Sigma(\mathcal{G}) &= \frac{1}{2}\sum_{A\in \mathsf{S}} OA^2 = \frac{1}{2}\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-1}\left(\left(n-\frac{1}{2}-i\right)^2+\left(n-\frac{1}{2}-j\right)^2\right) \\ &= \frac{1}{8}\cdot 4\cdot 2n\sum_{i=0}^{n-1}(2n-2i-1)^2 = n\sum_{j=0}^{n-1}(2j+1)^2 = n\left(\sum_{j=1}^{2n} j^2-\sum_{j=1}^{n}(2j)^2\right) \\ &= n\left(\frac{2n(2n+1)(4n+1)}{6}-4\cdot\frac{n(n+1)(2n+1)}{6}\right) = \frac{n^2(2n+1)(2n-1)}{3} = \Sigma(n).\end{aligned}$$

Comment. There are several variations of the above solution, also working for both versions of the problem. E.g., one may implement only the inequality $[OA_iA_{i+1}] \leqslant \frac{1}{2}OA_i \cdot OA_{i+1}$ to obtain

$$\Sigma(\mathcal{F}) \leqslant \frac{1}{2}\sum_{i=1}^{4n^2} OK_i \cdot OL_i,$$

where both (K_i) and (L_i) are permutations of all points in S. The right hand side can then be bounded from above by means of the rearrangement inequality; the bound is also achieved on the collection $\mathcal{G}$.

However, Version 2 seems to be more difficult than Version 1. First of all, the optimal model for this version is much less easy to guess, until one finds an idea for proving the upper bound. Moreover, Version 1 allows different solutions which do not seem to be generalized easily — such as Solution 2 below.

Solution 2, for Version 1. Let $\mathcal{F}$ be an accessible set of quadrilaterals. For every quadrilateral $ABCD$ in $\mathcal{F}$ write

$$[ABCD] = \frac{AC\cdot BD}{2}\sin\phi \leqslant \frac{AC^2+BD^2}{4}, \tag{3}$$

where ϕ is the angle between AC and BD. Applying this estimate to all members in $\mathcal{F}$ we obtain

$$\Sigma(\mathcal{F}) \leqslant \frac{1}{4}\sum_{i=1}^{2n^2} A_iB_i^2,$$

where $A_1, A_2, \ldots, A_{2n^2}, B_1, B_2, \ldots, B_{2n^2}$ is some permutation of S. For brevity, denote

$$f\big((A_i),(B_i)\big) := \sum_{i=1}^{2n^2} A_iB_i^2.$$

The rest of the solution is based on the following lemma.

Lemma 2. The maximal value of $f\big((A_i),(B_i)\big)$ over all permutations of S equals $\frac{4}{3}n^2(4n^2-1)$ and is achieved when A_i is symmetric to B_i with respect to O, for every $i = 1, 2, \ldots, 2n^2$.

Proof. Let $A_i = (p_i, q_i)$ and $B_i = (r_i, s_i)$, for $i = 1, 2, \ldots, 2n^2$. We have

$$f\big((A_i),(B_i)\big) = \sum_{i=1}^{2n^2}(p_i-r_i)^2 + \sum_{i=1}^{2n^2}(q_i-s_i)^2;$$

it suffices to bound the first sum, the second is bounded similarly. This can be done, e.g., by means of the QM–AM inequality as follows:

$$\begin{aligned}
\sum_{i=1}^{2n^2}(p_i-r_i)^2 &= \sum_{i=1}^{2n^2}\bigl(2p_i^2+2r_i^2-(p_i+r_i)^2\bigr) = 4n\sum_{j=0}^{2n-1} j^2 - \sum_{i=1}^{2n^2}(p_i+r_i)^2 \\
&\leqslant 4n\sum_{j=0}^{2n-1} j^2 - \frac{1}{2n^2}\left(\sum_{i=1}^{2n^2}(p_i+r_i)\right)^2 = 4n\sum_{j=0}^{2n-1} j^2 - \frac{1}{2n^2}\left(2n\cdot\sum_{j=0}^{2n-1} j\right)^2 \\
&= 4n\cdot\frac{2n(2n-1)(4n-1)}{6} - 2n^2(2n-1)^2 = \frac{2n^2(2n-1)(2n+1)}{3}.
\end{aligned}$$

All the estimates are sharp if $p_i + r_i = 2n - 1$ for all i. Thus,

$$f\bigl((A_i),(B_i)\bigr) \leqslant \frac{4n^2(4n^2-1)}{3},$$

and the estimate is sharp when $p_i + r_i = q_i + s_i = 2n - 1$ for all i, i.e. when A_i and B_i are symmetric with respect to O. $\square$

Lemma 2 yields

$$\Sigma(\mathcal{F}) \leqslant \frac{1}{4}\cdot\frac{4n^2(4n^2-1)}{3} = \frac{n^2(2n-1)(2n+1)}{3}.$$

Finally, all estimates are achieved simultaneously on the set $\mathcal{G}$ of central squares.

Comment 2. Lemma 2 also allows different proofs. E.g., one may optimize the sum $\sum_i p_i r_i$ step by step: if $p_i < p_j$ and $r_i < r_j$, then a swap $r_i \leftrightarrow r_j$ increases the sum. By applying a proper chain of such replacements (possibly swapping elements in some pairs (p_i, r_i)), one eventually comes to a permutation where $p_i + r_i = 2n - 1$ for all i.

Comment 3. Version 2 can also be considered for a square grid with odd number n of points on each side. If we allow a polygon consisting of one point, then Solution 1 is applied verbatim, providing an answer $\frac{1}{12}n^2(n^2-1)$. If such polygons are not allowed, then one needs to subtract $\frac{1}{2}$ from the answer.

G4. Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D intersect the rays BA and BC at points E and F, respectively. A point T is chosen inside the triangle ABC so that $TE \parallel CD$ and $TF \parallel AD$. Let $K \neq D$ be a point on the segment DF such that $TD = TK$. Prove that the lines AC, DT and BK intersect at one point.

Solution 1. Let the segments TE and TF cross AC at P and Q, respectively. Since $PE \parallel CD$ and ED is tangent to the circumcircle of $ABCD$, we have

$$\angle EPA = \angle DCA = \angle EDA,$$

and so the points A, P, D, and E lie on some circle α. Similarly, the points C, Q, D, and F lie on some circle γ.

We now want to prove that the line DT is tangent to both α and γ at D. Indeed, since $\angle FCD + \angle EAD = 180°$, the circles α and γ are tangent to each other at D. To prove that T lies on their common tangent line at D (i.e., on their radical axis), it suffices to check that $TP \cdot TE = TQ \cdot TF$, or that the quadrilateral $PEFQ$ is cyclic. This fact follows from

$$\angle QFE = \angle ADE = \angle APE.$$

Since $TD = TK$, we have $\angle TKD = \angle TDK$. Next, as TD and DE are tangent to α and Ω, respectively, we obtain

$$\angle TKD = \angle TDK = \angle EAD = \angle BDE,$$

which implies $TK \parallel BD$.

Next we prove that the five points T, P, Q, D, and K lie on some circle τ. Indeed, since TD is tangent to the circle α we have

$$\angle EPD = \angle TDF = \angle TKD,$$

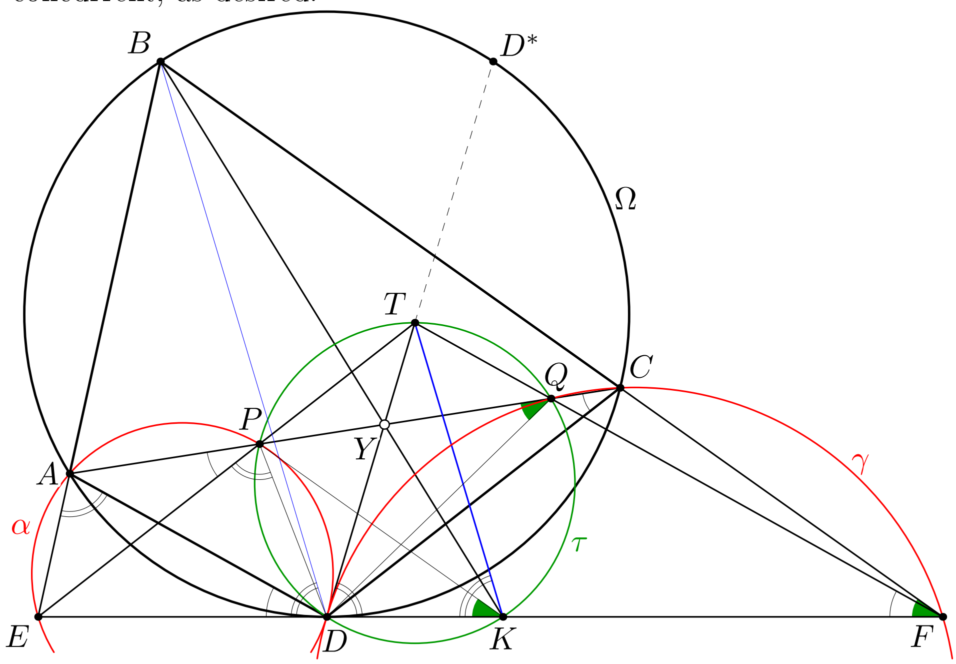
which means that the point P lies on the circle (TDK). Similarly, we have $Q \in (TDK)$.

Finally, we prove that $PK \parallel BC$. Indeed, using the circles τ and γ we conclude that

$$\angle PKD = \angle PQD = \angle DFC,$$

which means that $PK \parallel BC$.

Triangles TPK and DCB have pairwise parallel sides, which implies the fact that TD, PC and KB are concurrent, as desired.



Comment 1. There are several variations of the above solution.

E.g., after finding circles α and γ, one can notice that there exists a homothety h mapping the triangle TPQ to the triangle DCA; the centre of that homothety is $Y = AC \cap TD$. Since

$$\angle DPE = \angle DAE = \angle DCB = \angle DQT,$$

the quadrilateral $TPDQ$ is inscribed in some circle τ. We have $h(\tau) = \Omega$, so the point $D^* = h(D)$ lies on Ω.

Finally, by

$$\angle DCD^* = \angle TPD = \angle BAD,$$

the points B and D^* are symmetric with respect to the diameter of Ω passing through D. This yields $DB = DD^*$ and $BD^* \parallel EF$, so $h(K) = B$, and BK passes through Y.

Solution 2. Consider the spiral similarity ϕ centred at D which maps B to F. Recall that for any two points X and Y, the triangles $DX\phi(X)$ and $DY\phi(Y)$ are similar.

Define $T' = \phi(E)$. Then

$$\angle CDF = \angle FBD = \angle\phi(B)BD = \angle\phi(E)ED = \angle T'ED,$$

so $CD \parallel T'E$. Using the fact that DE is tangent to (ABD) and then applying ϕ we infer

$$\angle ADE = \angle ABD = \angle T'FD,$$

so $AD \parallel T'F$; hence T' coincides with T. Therefore,

$$\angle BDE = \angle FDT = \angle DKT,$$

whence $TK \parallel BD$.

Let $BK \cap TD = X$, $AC \cap TD = Y$, and $AC \cap TF = Q$. Notice that $TK \parallel BD$ implies

$$\frac{TX}{XD} = \frac{TK}{BD} = \frac{TD}{BD}.$$

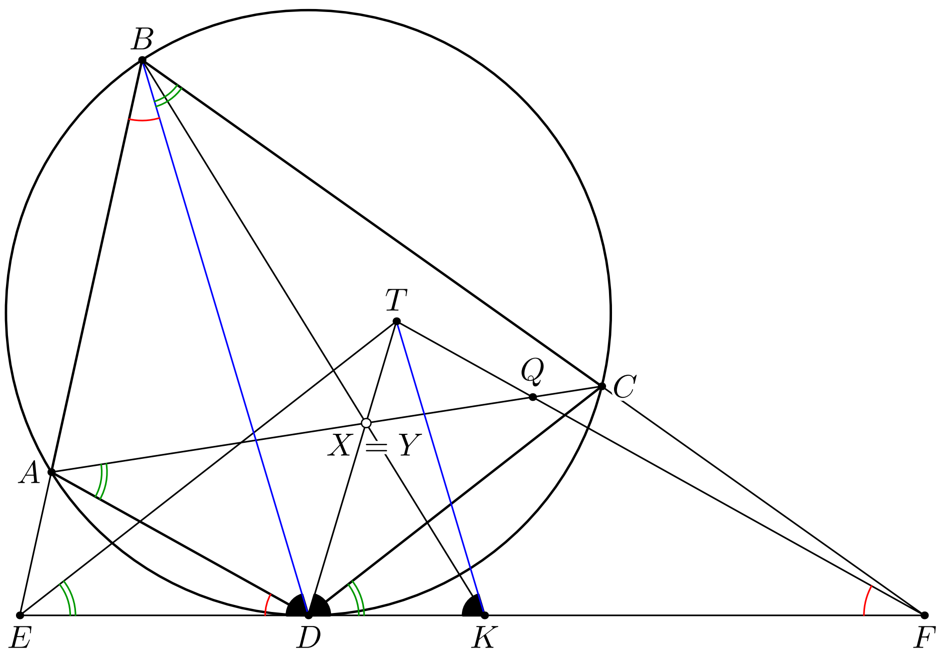
So we wish to prove that $\frac{TY}{YD}$ is equal to the same ratio.

We first show that $\phi(A) = Q$. Indeed,

$$\angle DA\phi(A) = \angle DBF = \angle DAC,$$

and so $\phi(A) \in AC$. Together with $\phi(A) \in \phi(EB) = TF$ this yields $\phi(A) = Q$. It follows that

$$\frac{TQ}{AE} = \frac{TD}{ED}.$$



Now, since $TF \parallel AD$ and $\triangle EAD \sim \triangle EDB$, we have

$$\frac{TY}{YD} = \frac{TQ}{AD} = \frac{TQ}{AE} \cdot \frac{AE}{AD} = \frac{TD}{ED} \cdot \frac{ED}{BD} = \frac{TD}{BD},$$

which completes the proof.

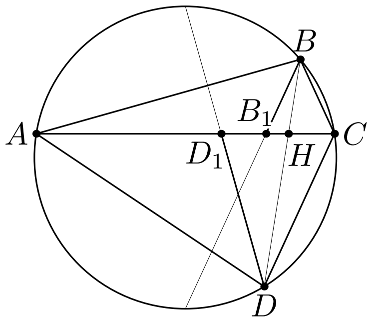
Comment 2. The point D is the Miquel point for any 4 of the 5 lines BA, BC, TE, TF and AC. Essentially, this is proved in both solutions by different methods.

G5. Let $ABCD$ be a cyclic quadrilateral whose sides have pairwise different lengths. Let O be the circumcentre of $ABCD$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at B_1 and D_1, respectively. Let O_B be the centre of the circle which passes through B and is tangent to AC at D_1. Similarly, let O_D be the centre of the circle which passes through D and is tangent to AC at B_1.

Assume that $BD_1 \parallel DB_1$. Prove that O lies on the line O_BO_D.

Common remarks. We introduce some objects and establish some preliminary facts common for all solutions below.

Let Ω denote the circle $(ABCD)$, and let γ_B and γ_D denote the two circles from the problem statement (their centres are O_B and O_D, respectively). Clearly, all three centres O, O_B, and O_D are distinct.



Assume, without loss of generality, that $AB > BC$. Suppose that $AD > DC$, and let $H = AC \cap BD$. Then the rays BB_1 and DD_1 lie on one side of BD, as they contain the midpoints of the arcs ADC and ABC, respectively. However, if $BD_1 \parallel DB_1$, then B_1 and D_1 should be separated by H. This contradiction shows that $AD < CD$.

Let γ_B and γ_D meet Ω again at T_B and T_D, respectively. The common chord BT_B of Ω and γ_B is perpendicular to their line of centres O_BO; likewise, $DT_D \perp O_DO$. Therefore, $O \in O_BO_D \iff O_BO \parallel O_DO \iff BT_B \parallel DT_D$, and the problem reduces to showing that

$$BT_B \parallel DT_D. \tag{1}$$

Comment 1. It seems that the discussion of the positions of points is necessary for both Solutions below. However, this part automatically follows from the angle chasing in Comment 2.

Solution 1. Let the diagonals AC and BD cross at H. Consider the homothety h centred at H and mapping B to D. Since $BD_1 \parallel DB_1$, we have $h(D_1) = B_1$.

Let the tangents to Ω at B and D meet AC at L_B and L_D, respectively. We have

$$\angle L_BBB_1 = \angle L_BBC + \angle CBB_1 = \angle BAL_B + \angle B_1BA = \angle BB_1L_B,$$

which means that the triangle L_BBB_1 is isosceles, $L_BB = L_BB_1$. The powers of L_B with respect to Ω and γ_D are L_BB^2 and $L_BB_1^2$, respectively; so they are equal, whence L_B lies on the radical axis T_DD of those two circles. Similarly, L_D lies on the radical axis T_BB of Ω and γ_B.

By the sine rule in the triangle BHL_B, we obtain

$$\frac{HL_B}{\sin\angle HBL_B} = \frac{BL_B}{\sin\angle BHL_B} = \frac{B_1L_B}{\sin\angle BHL_B}; \tag{2}$$

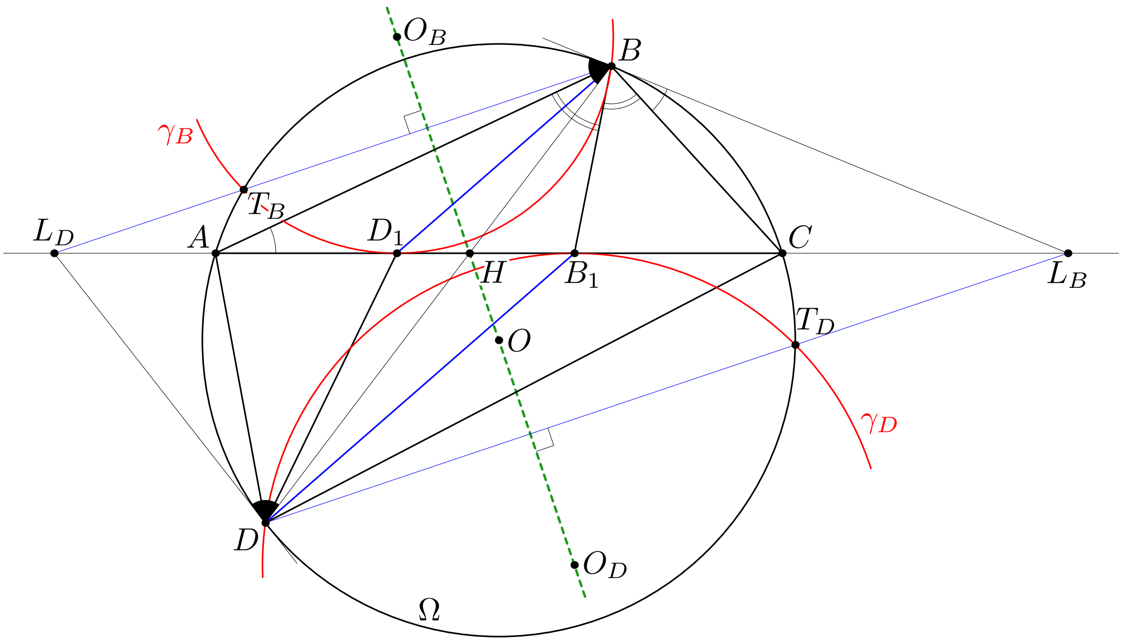
similarly,

$$\frac{HL_D}{\sin\angle HDL_D} = \frac{DL_D}{\sin\angle DHL_D} = \frac{D_1L_D}{\sin\angle DHL_D}. \tag{3}$$

Clearly, $\angle BHL_B = \angle DHL_D$. In the circle Ω, tangent lines BL_B and DL_D form equal angles with the chord BD, so $\sin\angle HBL_B = \sin\angle HDL_D$ (this equality does not depend on the picture). Thus, dividing (2) by (3) we get

$$\frac{HL_B}{HL_D} = \frac{B_1L_B}{D_1L_D}, \qquad \text{and hence} \qquad \frac{HL_B}{HL_D} = \frac{HL_B - B_1L_B}{HL_D - D_1L_D} = \frac{HB_1}{HD_1}.$$

Since $h(D_1) = B_1$, the obtained relation yields $h(L_D) = L_B$, so h maps the line L_DB to L_BD, and these lines are parallel, as desired.



Comment 2. In the solution above, the key relation $h(L_D) = L_B$ was obtained via a short computation in sines. Here we present an alternative, pure synthetical way of establishing that.

Let the external bisectors of $\angle ABC$ and $\angle ADC$ cross AC at B_2 and D_2, respectively; assume that $\widehat{AB} > \widehat{CB}$. In the right-angled triangle BB_1B_2, the point L_B is a point on the hypothenuse such that $L_BB_1 = L_BB$, so L_B is the midpoint of B_1B_2.

Since DD_1 is the internal angle bisector of $\angle ADC$, we have

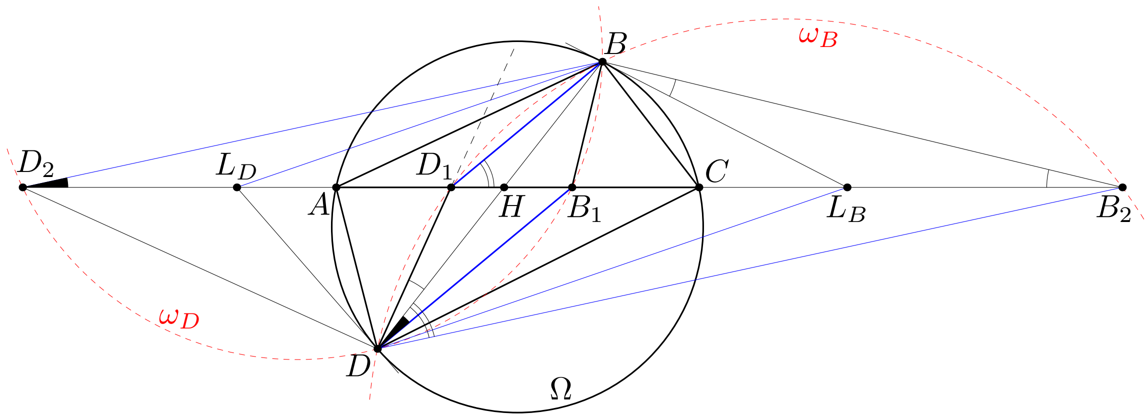
$$\angle BDD_1 = \frac{\angle BDA - \angle CDB}{2} = \frac{\angle BCA - \angle CAB}{2} = \angle BB_2D_1,$$

so the points B, B_2, D, and D_1 lie on some circle ω_B. Similarly, L_D is the midpoint of D_1D_2, and the points D, D_2, B, and B_1 lie on some circle ω_D.

Now we have

$$\angle B_2DB_1 = \angle B_2DB - \angle B_1DB = \angle B_2D_1B - \angle B_1D_2B = \angle D_2BD_1.$$

Therefore, the corresponding sides of the triangles DB_1B_2 and BD_1D_2 are parallel, and the triangles are homothetical (in H). So their corresponding medians DL_B and BL_D are also parallel.



Yet alternatively, after obtaining the circles ω_B and ω_D, one may notice that H lies on their radical axis BD, whence $HB_1 \cdot HD_2 = HD_1 \cdot HB_2$, or

$$\frac{HB_1}{HD_1} = \frac{HB_2}{HD_1}.$$

Since $h(D_1) = B_1$, this yields $h(D_2) = B_2$ and hence $h(L_D) = L_B$.

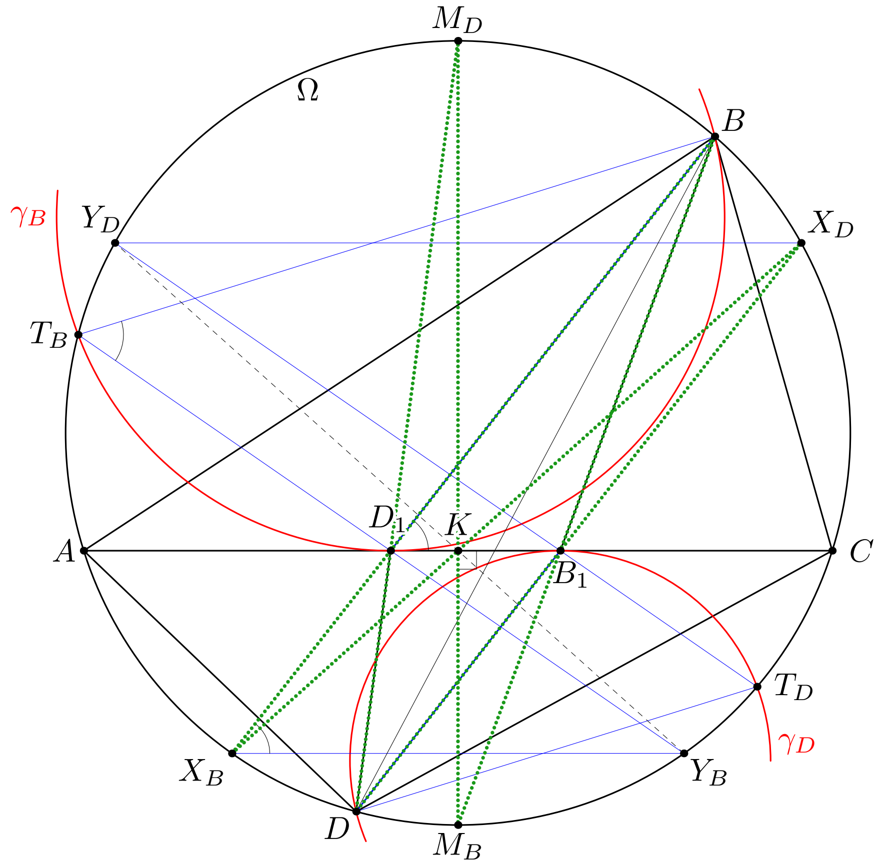
Comment 3. Since h preserves the line AC and maps $B \mapsto D$ and $D_1 \mapsto B_1$, we have $h(\gamma_B) = \gamma_D$. Therefore, $h(O_B) = O_D$; in particular, H also lies on O_BO_D.

Solution 2. Let BD_1 and T_BD_1 meet Ω again at X_B and Y_B, respectively. Then

$$\angle BD_1C = \angle BT_BD_1 = \angle BT_BY_B = \angle BX_BY_B,$$

which shows that $X_BY_B \parallel AC$. Similarly, let DB_1 and T_DB_1 meet Ω again at X_D and Y_D, respectively; then $X_DY_D \parallel AC$.

Let M_D and M_B be the midpoints of the arcs ABC and ADC, respectively; then the points D_1 and B_1 lie on DM_D and BM_B, respectively. Let K be the midpoint of AC (which lies on M_BM_D). Applying Pascal's theorem to $M_DDX_DX_BBM_B$, we obtain that the points $D_1 = M_DD \cap X_BB$, $B_1 = DX_D \cap BM_B$, and $X_DX_B \cap M_BM_D$ are collinear, which means that X_BX_D passes through K. Due to symmetry, the diagonals of an isosceles trapezoid $X_BY_BX_DY_D$ cross at K.



Let b and d denote the distances from the lines X_BY_B and X_DY_D, respectively, to AC. Then we get

$$\frac{X_BY_B}{X_DY_D} = \frac{b}{d} = \frac{D_1X_B}{B_1X_D},$$

where the second equation holds in view of $D_1X_B \parallel B_1X_D$. Therefore, the triangles $D_1X_BY_B$ and $B_1X_DY_D$ are similar. The triangles D_1T_BB and B_1T_DD are similar to them and hence to each other. Since $BD_1 \parallel DB_1$, these triangles are also homothetical. This yields $BT_B \parallel DT_D$, as desired.

Comment 4. The original problem proposal asked to prove that the relations $BD_1 \parallel DB_1$ and $O \in O_1O_2$ are equivalent. After obtaining $BD_1 \parallel DB_1 \Rightarrow O \in O_1O_2$, the converse proof is either repeated backwards *mutatis mutandis*, or can be obtained by the usual procedure of varying some points in the construction.

The Problem Selection Committee chose the current version, because it is less technical, yet keeps most of the ideas.

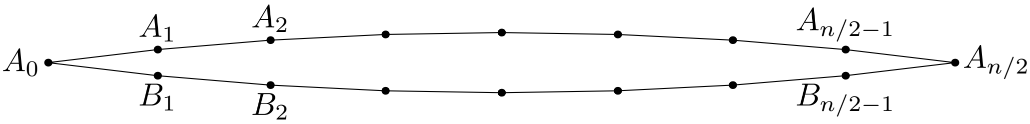
This page is intentionally left blank

G6. Determine all integers $n \geqslant 3$ satisfying the following property: every convex n-gon whose sides all have length 1 contains an equilateral triangle of side length 1.

(Every polygon is assumed to contain its boundary.)

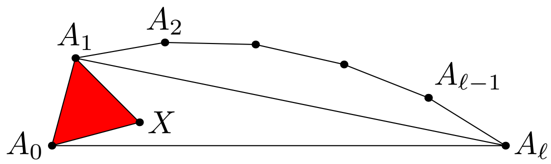
Answer: All odd $n \geqslant 3$.

Solution. First we show that for every even $n \geqslant 4$ there exists a polygon violating the required statement. Consider a regular k-gon $A_0A_1, \ldots A_{k-1}$ with side length 1. Let $B_1, B_2, \ldots, B_{n/2-1}$ be the points symmetric to $A_1, A_2, \ldots, A_{n/2-1}$ with respect to the line $A_0A_{n/2}$. Then $P = A_0A_1A_2 \ldots A_{n/2-1}A_{n/2}B_{n/2-1}B_{n/2-2} \ldots B_2B_1$ is a convex n-gon whose sides all have length 1. If k is big enough, P is contained in a strip of width $1/2$, which clearly does not contain any equilateral triangle of side length 1.



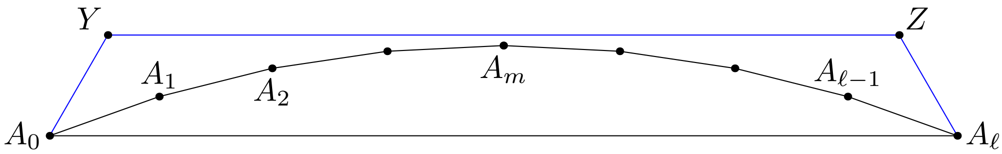
Assume now that $n = 2k+1$. As the case $k = 1$ is trivially true, we assume $k \geqslant 2$ henceforth. Consider a convex $(2k+1)$-gon P whose sides all have length 1. Let d be its longest diagonal. The endpoints of d split the perimeter of P into two polylines, one of which has length at least $k+1$. Hence we can label the vertices of P so that $P = A_0A_1 \ldots A_{2k}$ and $d = A_0A_\ell$ with $\ell \geqslant k+1$. We will show that, in fact, the polygon $A_0A_1 \ldots A_\ell$ contains an equilateral triangle of side length 1.

Suppose that $\angle A_\ell A_0 A_1 \geqslant 60^\circ$. Since d is the longest diagonal, we have $A_1A_\ell \leqslant A_0A_\ell$, so $\angle A_0A_1A_\ell \geqslant \angle A_\ell A_0 A_1 \geqslant 60^\circ$. It follows that there exists a point X inside the triangle $A_0A_1A_\ell$ such that the triangle A_0A_1X is equilateral, and this triangle is contained in P. Similar arguments apply if $\angle A_{\ell-1}A_\ell A_0 \geqslant 60^\circ$.



From now on, assume $\angle A_\ell A_0 A_1 < 60^\circ$ and $A_{\ell-1}A_\ell A_0 < 60^\circ$.

Consider an isosceles trapezoid A_0YZA_ℓ such that $A_0A_\ell \parallel YZ$, $A_0Y = ZA_\ell = 1$, and $\angle A_\ell A_0 Y = \angle ZA_\ell A_0 = 60^\circ$. Suppose that $A_0A_1 \ldots A_\ell$ is contained in A_0YZA_ℓ. Note that the perimeter of $A_0A_1 \ldots A_\ell$ equals $\ell + A_0A_\ell$ and the perimeter of A_0YZA_ℓ equals $2A_0A_\ell + 1$.



Recall a well-known fact stating that if a convex polygon P_1 is contained in a convex polygon P_2, then the perimeter of P_1 is at most the perimeter of P_2. Hence we obtain

$$\ell + A_0A_\ell \leqslant 2A_0A_\ell + 1, \quad \text{i.e.} \quad \ell - 1 \leqslant A_0A_\ell.$$

On the other hand, the triangle inequality yields

$$A_0A_\ell < A_\ell A_{\ell+1} + A_{\ell+1}A_{\ell+2} + \ldots + A_{2k}A_0 = 2k + 1 - \ell \leqslant \ell - 1,$$

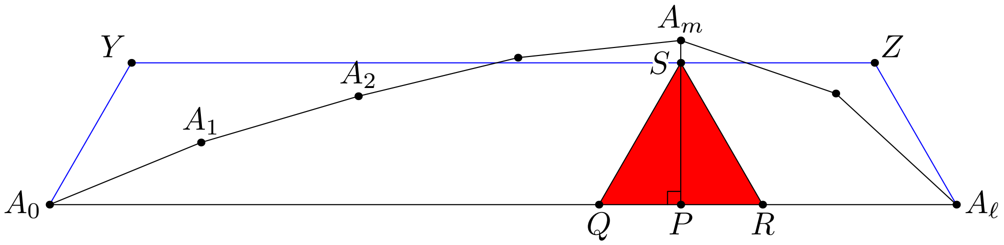
which gives a contradiction.

Therefore, there exists a vertex A_m of $A_0A_1 \ldots A_\ell$ which lies outside A_0YZA_ℓ. Since

$$\angle A_\ell A_0 A_1 < 60^\circ = \angle A_\ell A_0 Y \quad \text{and} \quad A_{\ell-1}A_\ell A_0 < 60^\circ = \angle ZA_\ell A_0, \tag{1}$$

the distance between A_m and A_0A_ℓ is at least $\sqrt{3}/2$.

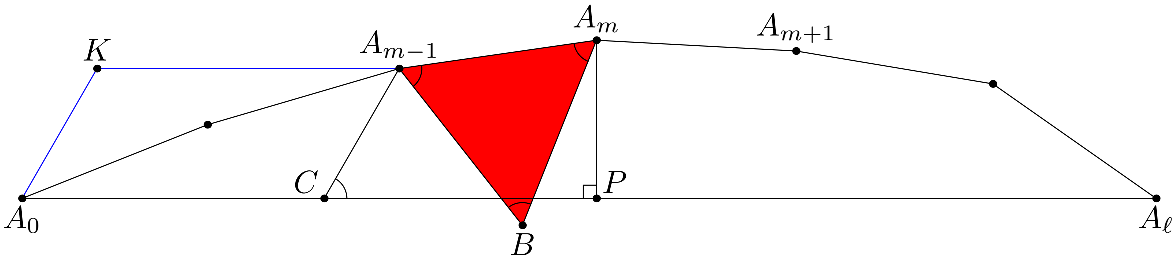
Let P be the projection of A_m to A_0A_ℓ. Then $PA_m \geqslant \sqrt{3}/2$, and by (1) we have $A_0P > 1/2$ and $PA_\ell > 1/2$. Choose points $Q \in A_0P$, $R \in PA_\ell$, and $S \in PA_m$ such that $PQ = PR = 1/2$ and $PS = \sqrt{3}/2$. Then QRS is an equilateral triangle of side length 1 contained in $A_0A_1 \ldots A_\ell$.



Comment. In fact, for every odd n a stronger statement holds, which is formulated in terms defined in the solution above: there exists an equilateral triangle $A_iA_{i+1}B$ contained in $A_0A_1 \ldots A_\ell$ for some $0 \leqslant i < \ell$. We sketch an indirect proof below.

As above, we get $\angle A_\ell A_0 A_1 < 60°$ and $A_{\ell-1}A_\ell A_0 < 60°$. Choose an index $m \in [1, \ell-1]$ maximising the distance between A_m and A_0A_ℓ. Arguments from the above solution yield $1 < m < \ell - 1$. Then $\angle A_0A_{m-1}A_m > 120°$ and $\angle A_{m-1}A_mA_\ell > \angle A_0A_mA_\ell \geqslant 60°$. We construct an equilateral triangle $A_{m-1}A_mB$ as in the figure below. If B lies in $A_0A_{m-1}A_mA_\ell$, then we are done. Otherwise B and A_m lie on different sides of A_0A_ℓ. As before, let P be the projection of A_m to A_0A_ℓ. We will show that

$$A_0A_1 + A_1A_2 + \ldots + A_{m-1}A_m < A_0P + 1/2. \tag{2}$$



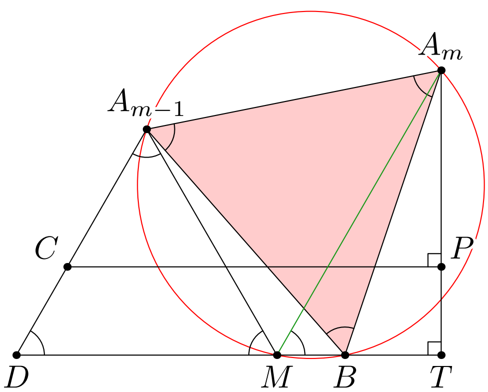
There exists a point C on the segment A_0P such that $\angle A_{m-1}CP = 60°$. Construct a parallelogram $A_0CA_{m-1}K$. Then the polyline $A_0A_1 \ldots A_{m-1}$ is contained in the triangle $A_{m-1}KA_0$, so

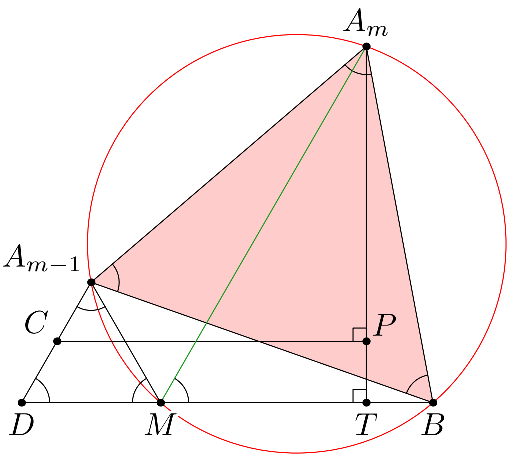
$$A_0A_1 + A_1A_2 + \ldots + A_{m-2}A_{m-1} + A_{m-1}A_m \leqslant A_0K + KA_{m-1} + A_{m-1}A_m = A_0C + CA_{m-1} + 1.$$

To prove (2), it suffices to show that $CA_{m-1} < CP - 1/2$. Let the line through B parallel to CP intersect the rays $A_{m-1}C$ and A_mP at D and T, respectively. It is easy to see that the desired inequality will follow from $DA_{m-1} \leqslant DT - 1/2$.

Two possible arrangements of points are shown in the figures below.

Observe that $\angle DA_{m-1}B \geqslant 60°$, so there is a point M on the segment DB such that the triangle DMA_{m-1} is equilateral. Then $\angle A_{m-1}MD = 60° = \angle A_{m-1}A_mB$, so $A_{m-1}MBA_m$ is a cyclic quadrilateral. Therefore, $\angle A_mMB = 60°$. Thus, T lies on the ray MB and we have to show that $MT \geqslant 1/2$. Indeed, $MT = A_mM/2$ and $A_mM \geqslant A_mB = 1$. This completes the proof of the inequality (2).





Similarly, either there exists an equilateral triangle $A_m A_{m+1} B'$ contained in $A_0 A_1 \ldots A_\ell$, or

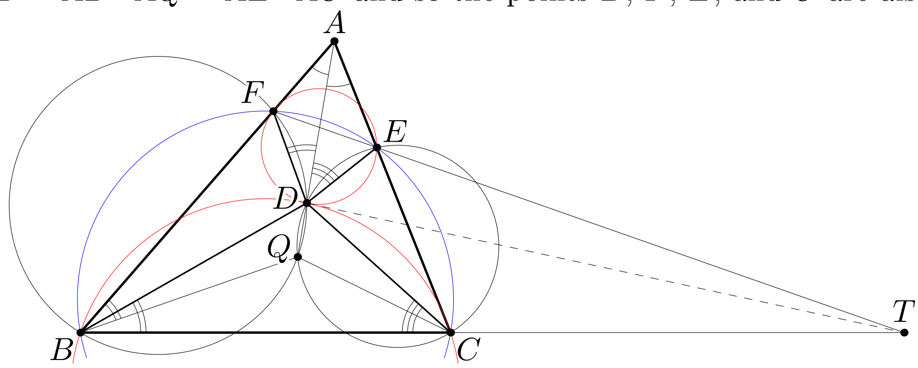
$$A_m A_{m+1} + A_{m+1} A_{m+2} + \ldots + A_{\ell-1} A_\ell < A_\ell P + 1/2. \tag{3}$$

Adding (2) and (3) yields $A_0 A_1 + A_1 A_2 + \ldots + A_{\ell-1} A_\ell < A_0 A_\ell + 1$, which gives a contradiction.

This page is intentionally left blank

G7. A point D is chosen inside an acute-angled triangle ABC with $AB > AC$ so that $\angle BAD = \angle DAC$. A point E is constructed on the segment AC so that $\angle ADE = \angle DCB$. Similarly, a point F is constructed on the segment AB so that $\angle ADF = \angle DBC$. A point X is chosen on the line AC so that $CX = BX$. Let O_1 and O_2 be the circumcentres of the triangles ADC and DXE. Prove that the lines BC, EF, and O_1O_2 are concurrent.

Common remarks. Let Q be the isogonal conjugate of D with respect to the triangle ABC. Since $\angle BAD = \angle DAC$, the point Q lies on AD. Then $\angle QBA = \angle DBC = \angle FDA$, so the points Q, D, F, and B are concyclic. Analogously, the points Q, D, E, and C are concyclic. Thus $AF \cdot AB = AD \cdot AQ = AE \cdot AC$ and so the points B, F, E, and C are also concyclic.



Let T be the intersection of BC and FE.

Claim. $TD^2 = TB \cdot TC = TF \cdot TE$.

Proof. We will prove that the circles (DEF) and (BDC) are tangent to each other. Indeed, using the above arguments, we get

$$\angle BDF = \angle AFD - \angle ABD = (180^\circ - \angle FAD - \angle FDA) - (\angle ABC - \angle DBC)$$
$$= 180^\circ - \angle FAD - \angle ABC = 180^\circ - \angle DAE - \angle FEA = \angle FED + \angle ADE = \angle FED + \angle DCB,$$

which implies the desired tangency.

Since the points B, C, E, and F are concyclic, the powers of the point T with respect to the circles (BDC) and (EDF) are equal. So their radical axis, which coincides with the common tangent at D, passes through T, and hence $TD^2 = TE \cdot TF = TB \cdot TC$. □

Solution 1. Let TA intersect the circle (ABC) again at M. Due to the circles $(BCEF)$ and $(AMCB)$, and using the above Claim, we get $TM \cdot TA = TF \cdot TE = TB \cdot TC = TD^2$; in particular, the points A, M, E, and F are concyclic.

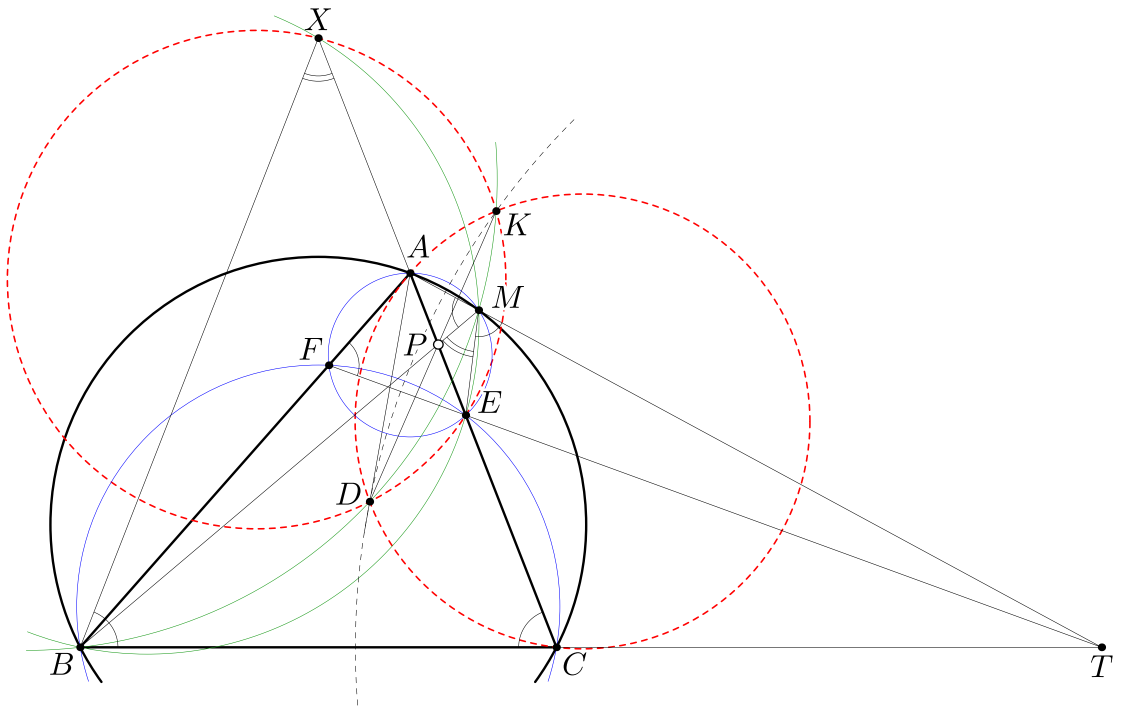
Under the inversion with centre T and radius TD, the point M maps to A, and B maps to C, which implies that the circle (MBD) maps to the circle (ADC). Their common point D lies on the circle of the inversion, so the second intersection point K also lies on that circle, which means $TK = TD$. It follows that the point T and the centres of the circles (KDE) and (ADC) lie on the perpendicular bisector of KD.

Since the center of (ADC) is O_1, it suffices to show now that the points D, K, E, and X are concyclic (the center of the corresponding circle will be O_2).

The lines BM, DK, and AC are the pairwise radical axes of the circles $(ABCM)$, $(ACDK)$ and $(BMDK)$, so they are concurrent at some point P. Also, M lies on the circle (AEF), thus

$$\measuredangle(EX, XB) = \measuredangle(CX, XB) = \measuredangle(XC, BC) + \measuredangle(BC, BX) = 2\measuredangle(AC, CB)$$
$$= \measuredangle(AC, CB) + \measuredangle(EF, FA) = \measuredangle(AM, BM) + \measuredangle(EM, MA) = \measuredangle(EM, BM),$$

so the points M, E, X, and B are concyclic. Therefore, $PE \cdot PX = PM \cdot PB = PK \cdot PD$, so the points E, K, D, and X are concyclic, as desired.



Comment 1. We present here a different solution which uses similar ideas.

Perform the inversion ι with centre T and radius TD. It swaps B with C and E with F; the point D maps to itself. Let $X' = \iota(X)$. Observe that the points E, F, X, and X' are concyclic, as well as the points B, C, X, and X'. Then

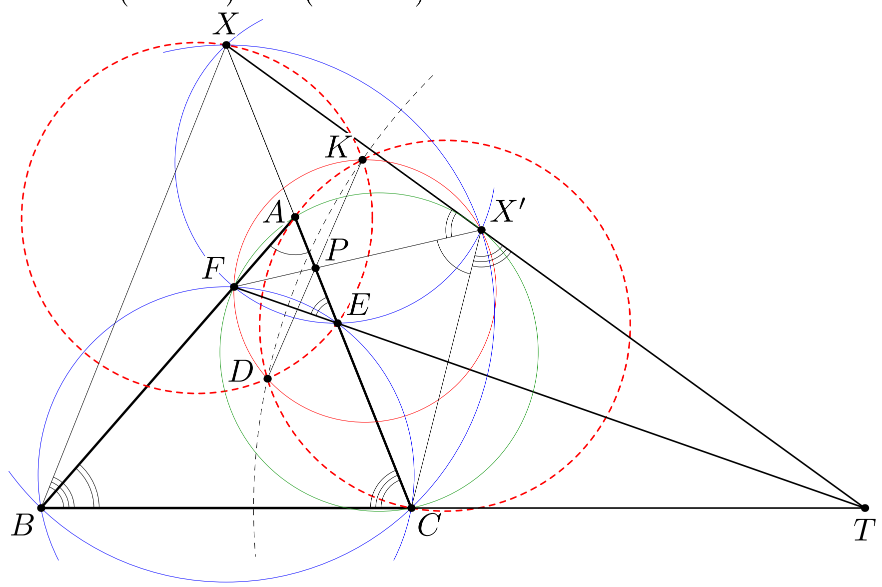
$$\begin{aligned}\measuredangle(CX', X'F) &= \measuredangle(CX', X'X) + \measuredangle(X'X, X'F) = \measuredangle(CB, BX) + \measuredangle(EX, EF)\\ &= \measuredangle(XC, CB) + \measuredangle(EC, EF) = \measuredangle(CA, CB) + \measuredangle(BC, BF) = \measuredangle(CA, AF),\end{aligned}$$

therefore the points C, X', A, and F are concyclic.

Let $X'F$ intersect AC at P, and let K be the second common point of DP and the circle (ACD). Then

$$PK \cdot PD = PA \cdot PC = PX' \cdot PF = PE \cdot PX;$$

hence, the points K, X, D, and E lie on some circle ω_1, while the points K, X', D, and F lie on some circle ω_2. (These circles are distinct since $\angle EXF + \angle EDF < \angle EAF + \angle DCB + \angle DBC < 180°$). The inversion ι swaps ω_1 with ω_2 and fixes their common point D, so it fixes their second common point K. Thus $TD = TK$ and the perpendicular bisector of DK passes through T, as well as through the centres of the circles $(CDKA)$ and $(DEKX)$.



Solution 2. We use only the first part of the Common remarks, namely, the facts that the tuples (C, D, Q, E) and (B, C, E, F) are both concyclic. We also introduce the point $T = BC \cap EF$. Let the circle (CDE) meet BC again at E_1. Since $\angle E_1CQ = \angle DCE$, the arcs DE and QE_1 of the circle (CDQ) are equal, so $DQ \parallel EE_1$.

Since $BFEC$ is cyclic, the line AD forms equal angles with BC and EF, hence so does EE_1. Therefore, the triangle EE_1T is isosceles, $TE = TE_1$, and T lies on the common perpendicular bisector of EE_1 and DQ.

Let U and V be the centres of circles (ADE) and $(CDQE)$, respectively. Then UO_1 is the perpendicular bisector of AD. Moreover, the points U, V, and O_2 belong to the perpendicular bisector of DE. Since $UO_1 \parallel VT$, in order to show that O_1O_2 passes through T, it suffices to show that

$$\frac{O_2U}{O_2V} = \frac{O_1U}{TV}. \tag{1}$$

Denote angles A, B, and C of the triangle ABC by α, β, and γ, respectively. Projecting onto AC we obtain

$$\frac{O_2U}{O_2V} = \frac{(XE - AE)/2}{(XE + EC)/2} = \frac{AX}{CX} = \frac{AX}{BX} = \frac{\sin(\gamma - \beta)}{\sin \alpha} \tag{2}$$

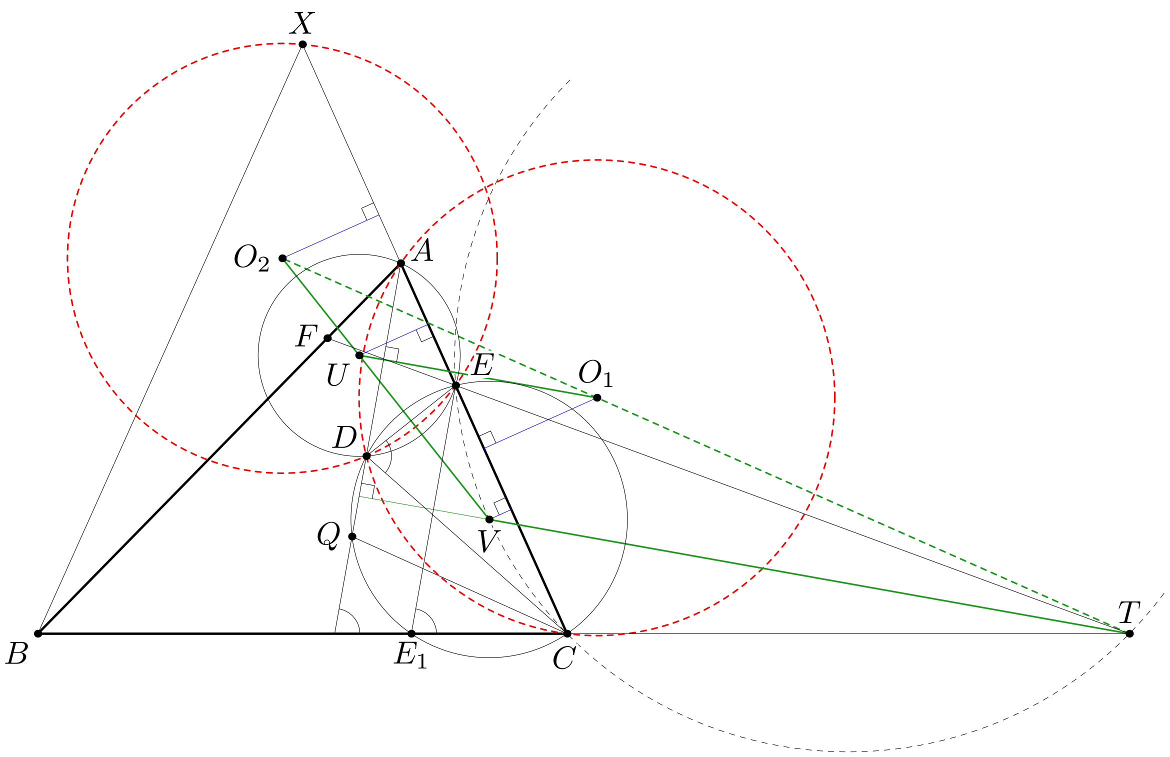
The projection of O_1U onto AC is $(AC - AE)/2 = CE/2$; the angle between O_1U and AC is $90° - \alpha/2$, so

$$\frac{O_1U}{EC} = \frac{1}{2\sin(\alpha/2)} \tag{3}$$

Next, we claim that E, V, C, and T are concyclic. Indeed, the point V lies on the perpendicular bisector of CE, as well as on the internal angle bisector of $\angle CTF$. Therefore, V coincides with the midpoint of the arc CE of the circle (TCE).

Now we have $\angle EVC = 2\angle EE_1C = 180° - (\gamma - \beta)$ and $\angle VET = \angle VE_1T = 90° - \angle E_1EC = 90° - \alpha/2$. Therefore,

$$\frac{EC}{TV} = \frac{\sin \angle ETC}{\sin \angle VET} = \frac{\sin(\gamma - \beta)}{\cos(\alpha/2)}. \tag{4}$$



Recalling (2) and multiplying (3) and (4) we establish (1):

$$\frac{O_2U}{O_2V} = \frac{\sin(\gamma-\beta)}{\sin\alpha} = \frac{1}{2\sin(\alpha/2)} \cdot \frac{\sin(\gamma-\beta)}{\cos(\alpha/2)} = \frac{O_1U}{EC} \cdot \frac{EC}{TV} = \frac{O_1U}{TV}.$$

Solution 3. Notice that $\angle AQE = \angle QCB$ and $\angle AQF = \angle QBC$; so, if we replace the point D with Q in the problem set up, the points E, F, and T remain the same. So, by the Claim, we have $TQ^2 = TB \cdot TC = TD^2$.

Thus, there exists a circle Γ centred at T and passing through D and Q. We denote the second meeting point of the circles Γ and (ADC) by K. Let the line AC meet the circle (DEK) again at Y; we intend to prove that $Y = X$. As in Solution 1, this will yield that the point T, as well as the centres O_1 and O_2, all lie on the perpendicular bisector of DK.

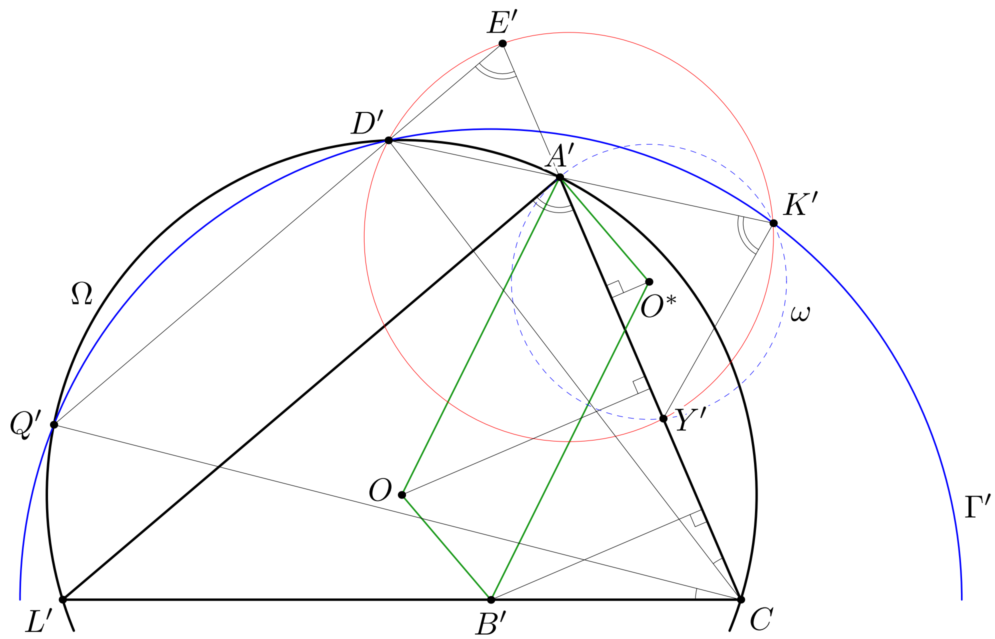
Let $L = AD \cap BC$. We perform an inversion centred at C; the images of the points will be denoted by primes, e.g., A' is the image of A. We obtain the following configuration, constructed in a triangle $A'CL'$.

The points D' and Q' are chosen on the circumcircle Ω of $A'L'C$ such that $\measuredangle(L'C, D'C) = \measuredangle(Q'C, A'C)$, which means that $A'L' \parallel D'Q'$. The lines $D'Q'$ and $A'C$ meet at E'.

A circle Γ' centred on CL' passes through D' and Q'. Notice here that B' lies on the segment CL', and that $\angle A'B'C = \angle BAC = 2\angle LAC = 2\angle A'L'C$, so that $B'L' = B'A'$, and B' lies on the perpendicular bisector of $A'L'$ (which coincides with that of $D'Q'$). All this means that B' is the centre of Γ'.

Finally, K' is the second meeting point of $A'D'$ and Γ', and Y' is the second meeting point of the circle $(D'K'E')$ and the line $A'E'$, We have $\measuredangle(Y'K', K'A') = \measuredangle(Y'E', E'D') = \measuredangle(Y'A', A'L')$, so $A'L'$ is tangent to the circumcircle ω of the triangle $Y'A'K'$.

Let O and O^* be the centres of Ω and ω, respectively. Then $O^*A' \perp A'L' \perp B'O$. The projections of vectors $\overrightarrow{O^*A'}$ and $\overrightarrow{B'O}$ onto $K'D'$ are equal to $\overrightarrow{K'A'}/2 = \overrightarrow{K'D'}/2 - \overrightarrow{A'D'}/2$. So $\overrightarrow{O^*A'} = \overrightarrow{B'O}$, or equivalently $\overrightarrow{A'O} = \overrightarrow{O^*B'}$. Projecting this equality onto $A'C$, we see that the projection of $\overrightarrow{O^*B'}$ equals $\overrightarrow{A'C}/2$. Since O^* is projected to the midpoint of $A'Y'$, this yields that B' is projected to the midpoint of CY', i.e., $B'Y' = B'C$ and $\angle B'Y'C = \angle B'CY'$. In the original figure, this rewrites as $\angle CBY = \angle BCY$, so Y lies on the perpendicular bisector of BC, as desired.



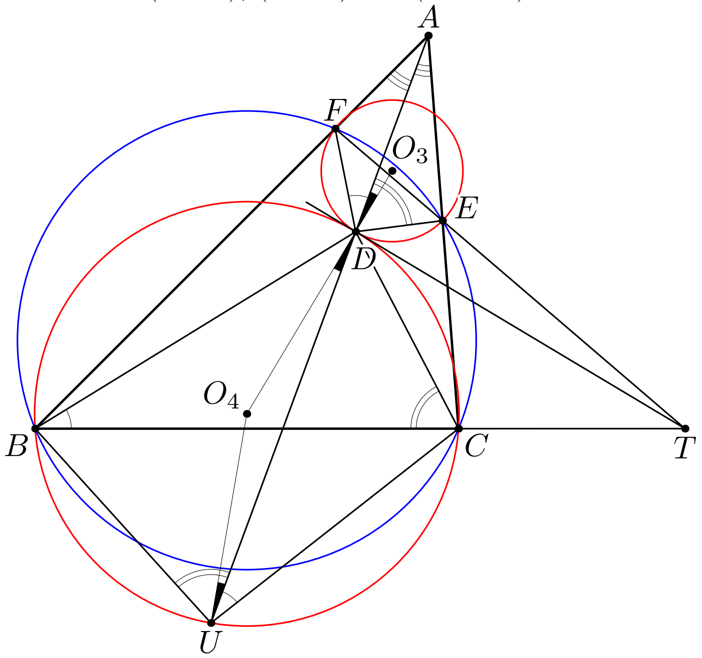
Comment 2. The point K appears to be the same in Solutions 1 and 3 (and Comment 1 as well). One can also show that K lies on the circle passing through A, X, and the midpoint of the arc BAC.

Comment 3. There are different proofs of the facts from the Common remarks, namely, the cyclicity of B, C, E, and F, and the Claim. We present one such alternative proof here.

We perform the composition ϕ of a homothety with centre A and the reflection in AD, which maps E to B. Let $U = \phi(D)$. Then $\measuredangle(BC, CD) = \measuredangle(AD, DE) = \measuredangle(BU, UD)$, so the points B, U, C, and D are concyclic. Therefore, $\measuredangle(CU, UD) = \measuredangle(CB, BD) = \measuredangle(AD, DF)$, so $\phi(F) = C$. Then the coefficient of the homothety is $AC/AF = AB/AE$, and thus points C, E, F, and B are concyclic.

Denote the centres of the circles (EDF) and $(BUCD)$ by O_3 and O_4, respectively. Then $\phi(O_3) = O_4$, hence $\measuredangle(O_3D, DA) = -\measuredangle(O_4U, UA) = \measuredangle(O_4D, DA)$, whence the circle (BDC) is tangent to the circle (EDF).

Now, the radical axes of circles (DEF), (BDC) and $(BCEF)$ intersect at T, and the claim follows.



This suffices for Solution 1 to work. However, Solutions 2 and 3 need properties of point Q, established in Common remarks before Solution 1.

Comment 4. In the original problem proposal, the point X was hidden. Instead, a circle γ was constructed such that D and E lie on γ, and its center is collinear with O_1 and T. The problem requested to prove that, in a fixed triangle ABC, independently from the choice of D on the bisector of $\angle BAC$, all circles γ pass through a fixed point.

G8. Let ω be the circumcircle of a triangle ABC, and let Ω_A be its excircle which is tangent to the segment BC. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y, respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $AR \perp BC$.

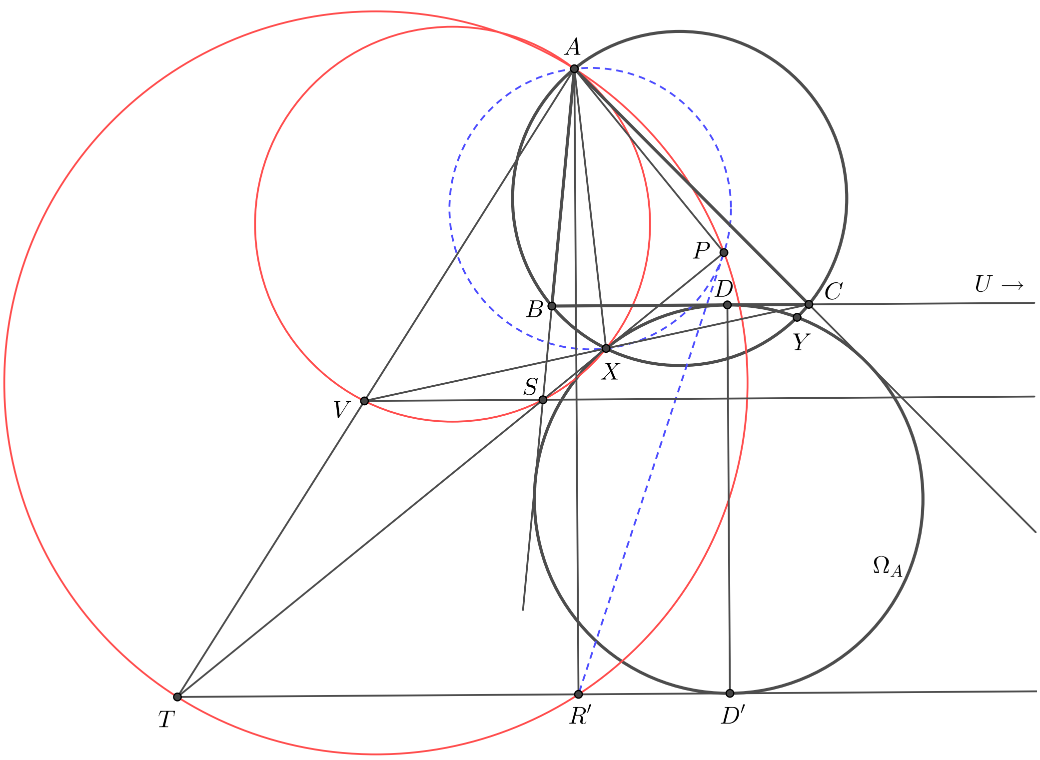
Solution 1. Let D be the point of tangency of BC and Ω_A. Let D' be the point such that DD' is a diameter of Ω_A. Let R' be (the unique) point such that $AR' \perp BC$ and $R'D' \parallel BC$. We shall prove that R' coincides with R.

Let PX intersect AB and $D'R'$ at S and T, respectively. Let U be the ideal common point of the parallel lines BC and $D'R'$. Note that the (degenerate) hexagon $ASXTUC$ is circumscribed around Ω_A, hence by the Brianchon theorem AT, SU, and XC concur at a point which we denote by V. Then $VS \parallel BC$. It follows that $\measuredangle(SV, VX) = \measuredangle(BC, CX) = \measuredangle(BA, AX)$, hence $AXSV$ is cyclic. Therefore, $\measuredangle(PX, XA) = \measuredangle(SV, VA) = \measuredangle(R'T, TA)$. Since $\angle APT = \angle AR'T = 90^\circ$, the quadrilateral $APR'T$ is cyclic. Hence,

$$\measuredangle(XA, AP) = 90^\circ - \measuredangle(PX, XA) = 90^\circ - \measuredangle(R'T, TA) = \measuredangle(TA, AR') = \measuredangle(TP, PR').$$

It follows that PR' is tangent to the circle (APX).

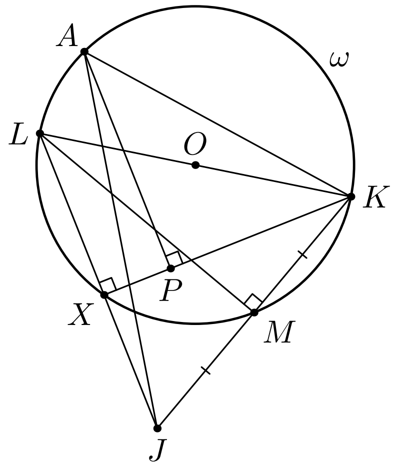
Analogous argument shows that QR' is tangent to the circle (AQY). Therefore, $R = R'$ and $AR \perp BC$.

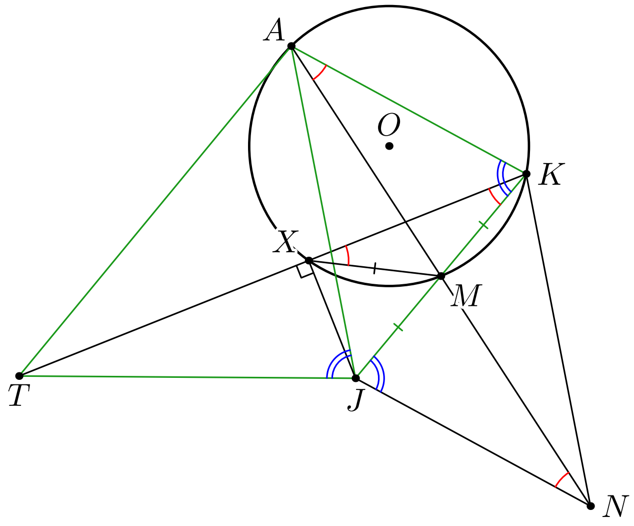


Comment 1. After showing $\measuredangle(PX, XA) = \measuredangle(R'T, TA)$ one can finish the solution as follows. There exists a spiral similarity mapping the triangle ATR' to the triangle AXP. So the triangles ATX and $AR'P$ are similar and equioriented. Thus, $\measuredangle(TX, XA) = \measuredangle(R'P, PA)$, which implies that PR' is tangent to the circle (APX).

Solution 2. Let J and r be the center and the radius of Ω_A. Denote the diameter of ω by d and its center by O. By Euler's formula, $OJ^2 = (d/2)^2 + dr$, so the power of J with respect to ω equals dr.

Let JX intersect ω again at L. Then $JL = d$. Let LK be a diameter of ω and let M be the midpoint of JK. Since $JL = LK$, we have $\angle LMK = 90^\circ$, so M lies on ω. Let R' be the point such that $R'P$ is tangent to the circle (APX) and $AR' \perp BC$. Note that the line AR' is symmetric to the line AO with respect to AJ.



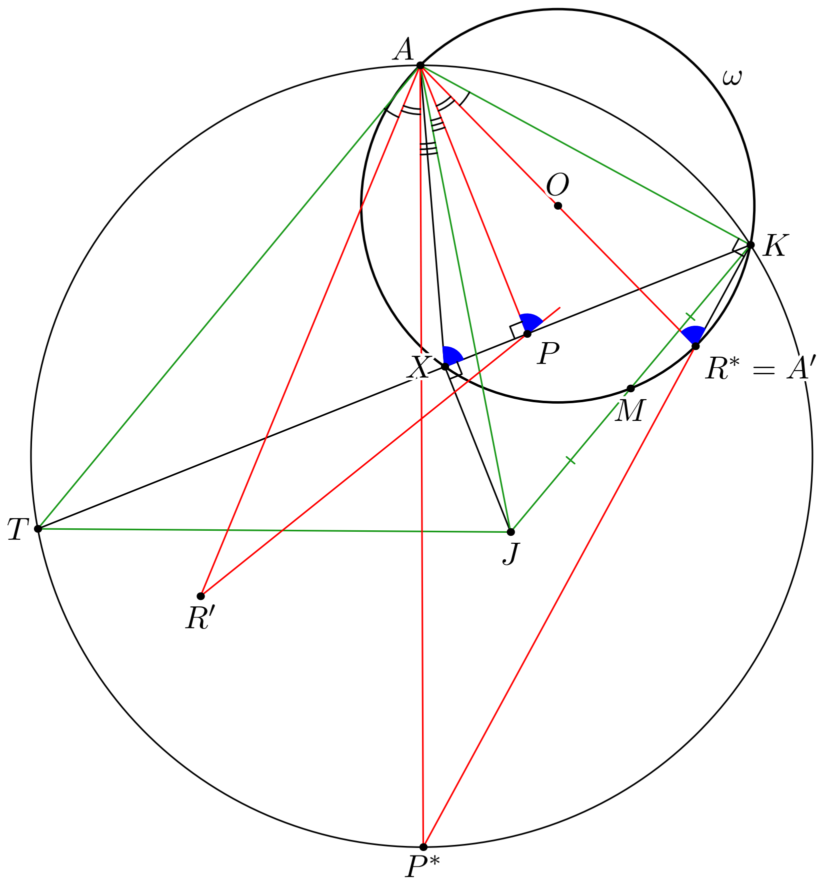


Lemma. Let M be the midpoint of the side JK in a triangle AJK. Let X be a point on the circle (AMK) such that $\angle JXK = 90^\circ$. Then there exists a point T on the line KX such that the triangles AKJ and AJT are similar and equioriented.

Proof. Note that $MX = MK$. We construct a parallelogram $AJNK$. Let T be a point on KX such that $\measuredangle(NJ, JA) = \measuredangle(KJ, JT)$. Then

$$\measuredangle(JN, NA) = \measuredangle(KA, AM) = \measuredangle(KX, XM) = \measuredangle(MK, KX) = \measuredangle(JK, KT).$$

So there exists a spiral similarity with center J mapping the triangle AJN to the triangle TJK. Therefore, the triangles NJK and AJT are similar and equioriented. It follows that the triangles AKJ and AJT are similar and equioriented. □



Back to the problem, we construct a point T as in the lemma. We perform the composition ϕ of inversion with centre A and radius AJ and reflection in AJ. It is known that every triangle AEF is similar and equioriented to $A\phi(F)\phi(E)$.

So $\phi(K) = T$ and $\phi(T) = K$. Let $P^* = \phi(P)$ and $R^* = \phi(R')$. Observe that $\phi(TK)$ is a circle with diameter AP^*. Let AA' be a diameter of ω. Then $P^*K \perp AK \perp A'K$, so A' lies on P^*K. The triangles $AR'P$ and AP^*R^* are similar and equioriented, hence

$$\measuredangle(AA', A'P^*) = \measuredangle(AA', A'K) = \measuredangle(AX, XP) = \measuredangle(AX, XP) = \measuredangle(AP, PR') = \measuredangle(AR^*, R^*P^*),$$

so A, A', R^*, and P^* are concyclic. Since A' and R^* lie on AO, we obtain $R^* = A'$. So $R' = \phi(A')$, and $\phi(A')P$ is tangent to the circle (APX).

An identical argument shows that $\phi(A')Q$ is tangent to the circle (AQY). Therefore, $R = \phi(A')$ and $AR \perp BC$.

Comment 2. One of the main ideas of Solution 2 is to get rid of the excircle, along with points B and C. After doing so we obtain the following fact, which is, essentially, proved in Solution 2.

Let ω be the circumcircle of a triangle AK_1K_2. Let J be a point such that the midpoints of JK_1 and JK_2 lie on ω. Points X and Y are chosen on ω so that $\angle JXK_1 = \angle JYK_2 = 90°$. Let P and Q be the projections of A onto XK_1 and YK_2, respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Then the reflection of the line AR in AJ passes through the centre O of ω.

Number Theory

N1. Determine all integers $n \geqslant 1$ for which there exists a pair of positive integers (a, b) such that no cube of a prime divides $a^2 + b + 3$ and

$$\frac{ab + 3b + 8}{a^2 + b + 3} = n.$$

Answer: The only integer with that property is $n = 2$.

Solution. As $b \equiv -a^2 - 3 \pmod{a^2 + b + 3}$, the numerator of the given fraction satisfies

$$ab + 3b + 8 \equiv a(-a^2 - 3) + 3(-a^2 - 3) + 8 \equiv -(a+1)^3 \pmod{a^2 + b + 3}.$$

As $a^2 + b + 3$ is not divisible by p^3 for any prime p, if $a^2 + b + 3$ divides $(a+1)^3$ then it does also divide $(a+1)^2$. Since

$$0 < (a+1)^2 < 2(a^2 + b + 3),$$

we conclude $(a+1)^2 = a^2 + b + 3$. This yields $b = 2(a-1)$ and $n = 2$. The choice $(a, b) = (2, 2)$ with $a^2 + b + 3 = 9$ shows that $n = 2$ indeed is a solution.

N2. Let $n \geqslant 100$ be an integer. The numbers $n, n+1, \ldots, 2n$ are written on $n+1$ cards, one number per card. The cards are shuffled and divided into two piles. Prove that one of the piles contains two cards such that the sum of their numbers is a perfect square.

Solution. To solve the problem it suffices to find three squares and three cards with numbers a, b, c on them such that pairwise sums $a+b, b+c, a+c$ are equal to the chosen squares. By choosing the three consecutive squares $(2k-1)^2$, $(2k)^2$, $(2k+1)^2$ we arrive at the triple

$$(a, b, c) = \left(2k^2 - 4k, \quad 2k^2 + 1, \quad 2k^2 + 4k\right).$$

We need a value for k such that

$$n \leqslant 2k^2 - 4k, \quad \text{and} \quad 2k^2 + 4k \leqslant 2n.$$

A concrete k is suitable for all n with

$$n \in \left[k^2 + 2k, 2k^2 - 4k + 1\right] =: I_k.$$

For $k \geqslant 9$ the intervals I_k and I_{k+1} overlap because

$$(k+1)^2 + 2(k+1) \leqslant 2k^2 - 4k + 1.$$

Hence $I_9 \cup I_{10} \cup \ldots = [99, \infty)$, which proves the statement for $n \geqslant 99$.

Comment 1. There exist approaches which only work for sufficiently large n.

One possible approach is to consider three cards with numbers $70k^2, 99k^2, 126k^2$ on them. Then their pairwise sums are perfect squares and so it suffices to find k such that $70k^2 \geqslant n$ and $126k^2 \leqslant 2n$ which exists for sufficiently large n.

Another approach is to prove, arguing by contradiction, that a and $a-2$ are in the same pile provided that n is large enough and a is sufficiently close to n. For that purpose, note that every pair of neighbouring numbers in the sequence $a, x^2-a, a+(2x+1), x^2+2x+3-a, a-2$ adds up to a perfect square for any x; so by choosing $x = \lfloor\sqrt{2a}\rfloor + 1$ and assuming that n is large enough we conclude that a and $a-2$ are in the same pile for any $a \in [n+2, 3n/2]$. This gives a contradiction since it is easy to find two numbers from $[n+2, 3n/2]$ of the same parity which sum to a square.

It then remains to separately cover the cases of small n which appears to be quite technical.

Comment 2. An alternative formulation for this problem could ask for a proof of the statement for all $n > 10^6$. An advantage of this formulation is that some solutions, e.g. those mentioned in Comment 1 need not contain a technical part which deals with the cases of small n. However, the original formulation seems to be better because the bound it gives for n is almost sharp, see the next comment for details.

Comment 3. The statement of the problem is false for $n = 98$. As a counterexample, the first pile may contain the even numbers from 98 to 126, the odd numbers from 129 to 161, and the even numbers from 162 to 196.

N3. Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \ldots, d_k)$ such that for every $i = 1, 2, \ldots, k$, the number $d_1 + \cdots + d_i$ is a perfect square.

Answer: $n = 1$ and $n = 3$.

Solution. For $i = 1, 2, \ldots, k$ let $d_1 + \ldots + d_i = s_i^2$, and define $s_0 = 0$ as well. Obviously $0 = s_0 < s_1 < s_2 < \ldots < s_k$, so

$$s_i \geqslant i \quad \text{and} \quad d_i = s_i^2 - s_{i-1}^2 = (s_i + s_{i-1})(s_i - s_{i-1}) \geqslant s_i + s_{i-1} \geqslant 2i - 1. \tag{1}$$

The number 1 is one of the divisors $d_1, \ldots, d_k$ but, due to $d_i \geqslant 2i - 1$, the only possibility is $d_1 = 1$.

Now consider d_2 and $s_2 \geqslant 2$. By definition, $d_2 = s_2^2 - 1 = (s_2 - 1)(s_2 + 1)$, so the numbers $s_2 - 1$ and $s_2 + 1$ are divisors of n. In particular, there is some index j such that $d_j = s_2 + 1$.

Notice that

$$s_2 + s_1 = s_2 + 1 = d_j \geqslant s_j + s_{j-1}; \tag{2}$$

since the sequence $s_0 < s_1 < \ldots < s_k$ increases, the index j cannot be greater than 2. Hence, the divisors $s_2 - 1$ and $s_2 + 1$ are listed among d_1 and d_2. That means $s_2 - 1 = d_1 = 1$ and $s_2 + 1 = d_2$; therefore $s_2 = 2$ and $d_2 = 3$.

We can repeat the above process in general.

Claim. $d_i = 2i - 1$ and $s_i = i$ for $i = 1, 2, \ldots, k$.

Proof. Apply induction on i. The Claim has been proved for $i = 1, 2$. Suppose that we have already proved $d = 1$, $d_2 = 3$, $\ldots$, $d_i = 2i - 1$, and consider the next divisor d_{i+1}:

$$d_{i+1} = s_{i+1}^2 - s_i^2 = s_{i+1}^2 - i^2 = (s_{i+1} - i)(s_{i+1} + i).$$

The number $s_{i+1} + i$ is a divisor of n, so there is some index j such that $d_j = s_{i+1} + i$.

Similarly to (2), by (1) we have

$$s_{i+1} + s_i = s_{i+1} + i = d_j \geqslant s_j + s_{j-1}; \tag{3}$$

since the sequence $s_0 < s_1 < \ldots < s_k$ increases, (3) forces $j \leqslant i + 1$. On the other hand, $d_j = s_{i+1} + i > 2i > d_i > d_{i-1} > \ldots > d_1$, so $j \leqslant i$ is not possible. The only possibility is $j = i + 1$.

Hence,

$$s_{i+1} + i = d_{i+1} = s_{i+1}^2 - s_i^2 = s_{i+1}^2 - i^2;$$
$$s_{i+1}^2 - s_{i+1} = i(i + 1).$$

By solving this equation we get $s_{i+1} = i + 1$ and $d_{i+1} = 2i + 1$, that finishes the proof. $\square$

Now we know that the positive divisors of the number n are $1, 3, 5, \ldots, n-2, n$. The greatest divisor is $d_k = 2k - 1 = n$ itself, so n must be odd. The second greatest divisor is $d_{k-1} = n - 2$; then $n - 2$ divides $n = (n - 2) + 2$, so $n - 2$ divides 2. Therefore, n must be 1 or 3.

The numbers $n = 1$ and $n = 3$ obviously satisfy the requirements: for $n = 1$ we have $k = 1$ and $d_1 = 1^2$; for $n = 3$ we have $k = 2$, $d_1 = 1^2$ and $d_1 + d_2 = 1 + 3 = 2^2$.

This page is intentionally left blank

N4. Alice is given a rational number $r>1$ and a line with two points $B \neq R$, where point R contains a red bead and point B contains a blue bead. Alice plays a solitaire game by performing a sequence of moves. In every move, she chooses a (not necessarily positive) integer k, and a bead to move. If that bead is placed at point X, and the other bead is placed at Y, then Alice moves the chosen bead to point X' with $\overrightarrow{YX'} = r^k \overrightarrow{YX}$.

Alice's goal is to move the red bead to the point B. Find all rational numbers $r>1$ such that Alice can reach her goal in at most 2021 moves.

Answer: All $r=(b+1)/b$ with $b=1,\ldots,1010$.

Solution. Denote the red and blue beads by $\mathcal{R}$ and $\mathcal{B}$, respectively. Introduce coordinates on the line and identify the points with their coordinates so that $R=0$ and $B=1$. Then, during the game, the coordinate of $\mathcal{R}$ is always smaller than the coordinate of $\mathcal{B}$. Moreover, the distance between the beads always has the form r^ℓ with $\ell \in \mathbb{Z}$, since it only multiplies by numbers of this form. Denote the value of the distance after the m^{th} move by $d_m = r^{\alpha_m}$, $m=0,1,2,\ldots$ (after the 0^{th} move we have just the initial position, so $\alpha_0 = 0$).

If some bead is moved in two consecutive moves, then Alice could instead perform a single move (and change the distance from d_i directly to d_{i+2}) which has the same effect as these two moves. So, if Alice can achieve her goal, then she may as well achieve it in fewer (or the same) number of moves by alternating the moves of $\mathcal{B}$ and $\mathcal{R}$. In the sequel, we assume that Alice alternates the moves, and that $\mathcal{R}$ is shifted altogether t times.

If $\mathcal{R}$ is shifted in the m^{th} move, then its coordinate increases by $d_m - d_{m+1}$. Therefore, the total increment of $\mathcal{R}$'s coordinate, which should be 1, equals

$$\text{either} \qquad (d_0-d_1)+(d_2-d_3)+\cdots+(d_{2t-2}-d_{2t-1}) = 1+\sum_{i=1}^{t-1} r^{\alpha_{2i}} - \sum_{i=1}^{t} r^{\alpha_{2i-1}},$$

$$\text{or} \qquad (d_1-d_2)+(d_3-d_4)+\cdots+(d_{2t-1}-d_{2t}) = \sum_{i=1}^{t} r^{\alpha_{2i-1}} - \sum_{i=1}^{t} r^{\alpha_{2i}},$$

depending on whether $\mathcal{R}$ or $\mathcal{B}$ is shifted in the first move. Moreover, in the former case we should have $t \leqslant 1011$, while in the latter one we need $t \leqslant 1010$. So both cases reduce to an equation

$$\sum_{i=1}^{n} r^{\beta_i} = \sum_{i=1}^{n-1} r^{\gamma_i}, \qquad \beta_i, \gamma_i \in \mathbb{Z}, \tag{1}$$

for some $n \leqslant 1011$. Thus, if Alice can reach her goal, then this equation has a solution for $n=1011$ (we can add equal terms to both sums in order to increase n).

Conversely, if (1) has a solution for $n=1011$, then Alice can compose a corresponding sequence of distances $d_0, d_1, d_2, \ldots, d_{2021}$ and then realise it by a sequence of moves. So the problem reduces to the solvability of (1) for $n=1011$.

Assume that, for some rational r, there is a solution of (1). Write r in lowest terms as $r=a/b$. Substitute this into (1), multiply by the common denominator, and collect all terms on the left hand side to get

$$\sum_{i=1}^{2n-1} (-1)^i a^{\mu_i} b^{N-\mu_i} = 0, \qquad \mu_i \in \{0,1,\ldots,N\}, \tag{2}$$

for some $N \geqslant 0$. We assume that there exist indices j_- and j_+ such that $\mu_{j_-}=0$ and $\mu_{j_+}=N$.

Reducing (2) modulo $a-b$ (so that $a \equiv b$), we get

$$0=\sum_{i=1}^{2n-1}(-1)^i a^{\mu_i} b^{N-\mu_i} \equiv \sum_{i=1}^{2n-1}(-1)^i b^{\mu_i} b^{N-\mu_i}=-b^N \mod (a-b).$$

Since $\gcd(a-b, b)=1$, this is possible only if $a-b=1$.

Reducing (2) modulo $a+b$ (so that $a \equiv -b$), we get

$$0=\sum_{i=1}^{2n-1}(-1)^i a^{\mu_i} b^{N-\mu_i} \equiv \sum_{i=1}^{2n-1}(-1)^i(-1)^{\mu_i} b^{\mu_i} b^{N-\mu_i}=S b^N \mod (a+b)$$

for some odd (thus nonzero) S with $|S| \leqslant 2n-1$. Since $\gcd(a+b, b)=1$, this is possible only if $a+b \mid S$. So $a+b \leqslant 2n-1$, and hence $b=a-1 \leqslant n-1=1010$.

Thus we have shown that any sought r has the form indicated in the answer. It remains to show that for any $b=1,2, \ldots, 1010$ and $a=b+1$, Alice can reach the goal. For this purpose, in (1) we put $n=a$, $\beta_1=\beta_2=\cdots=\beta_a=0$, and $\gamma_1=\gamma_2=\cdots=\gamma_b=1$.

Comment 1. Instead of reducing modulo $a+b$, one can reduce modulo a and modulo b. The first reduction shows that the number of terms in (2) with $\mu_i=0$ is divisible by a, while the second shows that the number of terms with $\mu_i=N$ is divisible by b.

Notice that, in fact, $N>0$, as otherwise (2) contains an alternating sum of an odd number of equal terms, which is nonzero. Therefore, all terms listed above have different indices, and there are at least $a+b$ of them.

Comment 2. Another way to investigate the solutions of equation (1) is to consider the Laurent polynomial

$$L(x)=\sum_{i=1}^{n} x^{\beta_i}-\sum_{i=1}^{n-1} x^{\gamma_i}.$$

We can pick a sufficiently large integer d so that $P(x)=x^d L(x)$ is a polynomial in $\mathbb{Z}[x]$. Then

$$P(1)=1, \tag{3}$$

and

$$1 \leqslant|P(-1)| \leqslant 2021. \tag{4}$$

If $r=p/q$ with integers $p>q \geqslant 1$ is a rational number with the properties listed in the problem statement, then $P(p/q)=L(p/q)=0$. As $P(x)$ has integer coefficients,

$$(p-q x) \mid P(x). \tag{5}$$

Plugging $x=1$ into (5) gives $(p-q) \mid P(1)=1$, which implies $p=q+1$. Moreover, plugging $x=-1$ into (5) gives $(p+q) \mid P(-1)$, which, along with (4), implies $p+q \leqslant 2021$ and $q \leqslant 1010$. Hence $x=(q+1)/q$ for some integer q with $1 \leqslant q \leqslant 1010$.

N5. Prove that there are only finitely many quadruples (a,b,c,n) of positive integers such that
$$n! = a^{n-1}+b^{n-1}+c^{n-1}.$$

Solution. For fixed n there are clearly finitely many solutions; we will show that there is no solution with $n>100$. So, assume $n>100$. By the AM–GM inequality,
$$\begin{aligned} n! &= 2n(n-1)(n-2)(n-3)\cdot(3\cdot 4\cdots(n-4)) \\ &\leqslant 2(n-1)^4\left(\frac{3+\cdots+(n-4)}{n-6}\right)^{n-6} = 2(n-1)^4\left(\frac{n-1}{2}\right)^{n-6} < \left(\frac{n-1}{2}\right)^{n-1}, \end{aligned}$$
thus $a,b,c<(n-1)/2$.

For every prime p and integer $m\neq 0$, let $\nu_p(m)$ denote the p-adic valuation of m; that is, the greatest non-negative integer k for which p^k divides m. Legendre's formula states that
$$\nu_p(n!) = \sum_{s=1}^{\infty}\left\lfloor \frac{n}{p^s}\right\rfloor,$$
and a well-know corollary of this formula is that
$$\nu_p(n!) < \sum_{s=1}^{\infty}\frac{n}{p^s} = \frac{n}{p-1}. \qquad (\heartsuit)$$

If n is odd then a^{n-1},b^{n-1},c^{n-1} are squares, and by considering them modulo 4 we conclude that a, b and c must be even. Hence, $2^{n-1}\mid n!$ but that is impossible for odd n because $\nu_2(n!)=\nu_2((n-1)!)<n-1$ by $(\heartsuit)$.

From now on we assume that n is even. If all three numbers $a+b, b+c, c+a$ are powers of 2 then a,b,c have the same parity. If they all are odd, then $n! = a^{n-1}+b^{n-1}+c^{n-1}$ is also odd which is absurd. If all a,b,c are divisible by 4, this contradicts $\nu_2(n!)\leqslant n-1$. If, say, a is not divisible by 4, then $2a=(a+b)+(a+c)-(b+c)$ is not divisible by 8, and since all $a+b$, $b+c$, $c+a$ are powers of 2, we get that one of these sums equals 4, so two of the numbers of a,b,c are equal to 2. Say, $a=b=2$, then $c=2^r-2$ and, since $c\mid n!$, we must have $c\mid a^{n-1}+b^{n-1}=2^n$ implying $r=2$, and so $c=2$, which is impossible because $n!\equiv 0\not\equiv 3\cdot 2^{n-1} \pmod 5$.

So now we assume that the sum of two numbers among a,b,c, say $a+b$, is not a power of 2, so it is divisible by some odd prime p. Then $p\leqslant a+b<n$ and so $c^{n-1}=n!-(a^{n-1}+b^{n-1})$ is divisible by p. If p divides a and b, we get $p^{n-1}\mid n!$, contradicting $(\heartsuit)$. Next, using $(\heartsuit)$ and the Lifting the Exponent Lemma we get
$$\nu_p(1)+\nu_p(2)+\cdots+\nu_p(n) = \nu_p(n!) = \nu_p(n!-c^{n-1}) = \nu_p\left(a^{n-1}+b^{n-1}\right) = \nu_p(a+b)+\nu_p(n-1). \quad (\diamondsuit)$$

In view of $(\diamondsuit)$, no number of $1,2,\ldots,n$ can be divisible by p, except $a+b$ and $n-1>a+b$. On the other hand, $p|c$ implies that $p<n/2$ and so there must be at least two such numbers. Hence, there are two multiples of p among $1,2,\ldots,n$, namely $a+b=p$ and $n-1=2p$. But this is another contradiction because $n-1$ is odd. This final contradiction shows that there is no solution of the equation for $n>100$.

Comment 1. The original version of the problem asked to find all solutions to the equation. The solution to that version is not much different but is more technical.

Comment 2. To find all solutions we can replace the bound $a, b, c < (n-1)/2$ for all n with a weaker bound $a, b, c \leqslant n/2$ only for even n, which is a trivial application of AM–GM to the tuple $(2, 3, \ldots, n)$. Then we may use the same argument for odd n (it works for $n \geqslant 5$ and does not require any bound on a, b, c), and for even n the same solution works for $n \geqslant 6$ unless we have $a + b = n - 1$ and $2\nu_p(n-1) = \nu_p(n!)$. This is only possible for $p = 3$ and $n = 10$ in which case we can consider the original equation modulo 7 to deduce that $7 \mid abc$ which contradicts the fact that $7^9 > 10!$. Looking at $n \leqslant 4$ we find four solutions, namely,

$$(a, b, c, n) = (1, 1, 2, 3), (1, 2, 1, 3), (2, 1, 1, 3), (2, 2, 2, 4).$$

Comment 3. For sufficiently large n, the inequality $a, b, c < (n-1)/2$ also follows from Stirling's formula.

N6. Determine all integers $n \geqslant 2$ with the following property: every n pairwise distinct integers whose sum is not divisible by n can be arranged in some order $a_1, a_2, \ldots, a_n$ so that n divides $1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n$.

Answer: All odd integers and all powers of 2.

Solution. If $n = 2^k a$, where $a \geqslant 3$ is odd and k is a positive integer, we can consider a set containing the number $2^k + 1$ and $n - 1$ numbers congruent to 1 modulo n. The sum of these numbers is congruent to 2^k modulo n and therefore is not divisible by n; for any permutation $(a_1, a_2, \ldots, a_n)$ of these numbers

$$1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n \equiv 1 + \cdots + n \equiv 2^{k-1}a(2^k a + 1) \not\equiv 0 \pmod{2^k}$$

and *a fortiori* $1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n$ is not divisible by n.

From now on, we suppose that n is either odd or a power of 2. Let S be the given set of integers, and s be the sum of elements of S.

Lemma 1. If there is a permutation (a_i) of S such that (n, s) divides $\sum_{i=1}^{n} ia_i$, then there is a permutation (b_i) of S such that n divides $\sum_{i=1}^{n} ib_i$.

Proof. Let $r = \sum_{i=1}^{n} ia_i$. Consider the permutation (b_i) defined by $b_i = a_{i+x}$, where $a_{j+n} = a_j$. For this permutation, we have

$$\sum_{i=1}^{n} ib_i = \sum_{i=1}^{n} ia_{i+x} \equiv \sum_{i=1}^{n} (i - x)a_i \equiv r - sx \pmod{n}.$$

Since (n, s) divides r, the congruence $r - sx \equiv 0 \pmod{n}$ admits a solution.

Lemma 2. Every set T of km integers, $m > 1$, can be partitioned into m sets of k integers so that in every set either the sum of elements is not divisible by k or all the elements leave the same remainder upon division by k.

Proof. The base case, $m = 2$. If T contains k elements leaving the same remainder upon division by k, we form one subset A of these elements; the remaining elements form a subset B. If k does not divide the sum of all elements of B, we are done. Otherwise it is enough to exchange any element of A with any element of B not congruent to it modulo k, thus making sums of both A and B not divisible by k. This cannot be done only when all the elements of T are congruent modulo k; in this case any partition will do.

If no k elements of T have the same residue modulo k, there are three elements $a, b, c \in T$ leaving pairwise distinct remainders upon division by k. Let t be the sum of elements of T. It suffices to find $A \subset T$ such that $|A| = k$ and $\sum_{x \in A} x \not\equiv 0, t \pmod{k}$: then neither the sum of elements of A nor the sum of elements of $B = T \setminus A$ is divisible by k. Consider $U' \subset T \setminus \{a, b, c\}$ with $|U'| = k - 1$. The sums of elements of three sets $U' \cup \{a\}$, $U' \cup \{b\}$, $U' \cup \{c\}$ leave three different remainders upon division by k, and at least one of them is not congruent either to 0 or to t.

Now let $m > 2$. If T contains k elements leaving the same remainder upon division by k, we form one subset A of these elements and apply the inductive hypothesis to the remaining $k(m - 1)$ elements. Otherwise, we choose any $U \subset T$, $|U| = k - 1$. Since all the remaining elements cannot be congruent modulo k, there is $a \in T \setminus U$ such that $a \not\equiv -\sum_{x \in U} x \pmod{k}$. Now we can take $A = U \cup \{a\}$ and apply the inductive hypothesis to $T \setminus A$.

Now we are ready to prove the statement of the problem for all odd n and $n = 2^k$. The proof is by induction.

If n is prime, the statement follows immediately from Lemma 1, since in this case $(n, s) = 1$. Turning to the general case, we can find prime p and an integer t such that $p^t \mid n$ and $p^t \nmid s$. By Lemma 2, we can partition S into p sets of $\frac{n}{p} = k$ elements so that in every set either the sum of numbers is not divisible by k or all numbers have the same residue modulo k.

For sets in the first category, by the inductive hypothesis there is a permutation (a_i) such that $k \mid \sum_{i=1}^{k} ia_i$.

If n (and therefore k) is odd, then for each permutation (b_i) of a set in the second category we have

$$\sum_{i=1}^{k} ib_i \equiv b_1 \frac{k(k+1)}{2} \equiv 0 \pmod{k}.$$

By combining such permutation for all sets of the partition, we get a permutation (c_i) of S such that $k \mid \sum_{i=1}^{n} ic_i$. Since this sum is divisible by k, and k is divisible by (n, s), we are done by Lemma 1.

If $n = 2^s$, we have $p = 2$ and $k = 2^{s-1}$. Then for each of the subsets there is a permutation $(a_1, \dots, a_k)$ such that $\sum_{i=1}^{k} ia_i$ is divisible by $2^{s-2} = \frac{k}{2}$: if the subset belongs to the first category, the expression is divisible even by k, and if it belongs to the second one,

$$\sum_{i=1}^{k} ia_i \equiv a_1 \frac{k(k+1)}{2} \equiv 0 \left(\operatorname{mod} \frac{k}{2}\right).$$

Now the numbers of each permutation should be multiplied by all the odd or all the even numbers not exceeding n in increasing order so that the resulting sums are divisible by k:

$$\sum_{i=1}^{k} (2i-1)a_i \equiv \sum_{i=1}^{k} 2ia_i \equiv 2\sum_{i=1}^{k} ia_i \equiv 0 \pmod{k}.$$

Combining these two sums, we again get a permutation (c_i) of S such that $k \mid \sum_{i=1}^{n} ic_i$, and finish the case by applying Lemma 1.

Comment. We cannot dispense with the condition that n does not divide the sum of all elements. Indeed, for each $n > 1$ and the set consisting of 1, -1, and $n-2$ elements divisible by n the required permutation does not exist.

N7. Let $a_1, a_2, a_3, \dots$ be an infinite sequence of positive integers such that a_{n+2m} divides $a_n + a_{n+m}$ for all positive integers n and m. Prove that this sequence is eventually periodic, i.e. there exist positive integers N and d such that $a_n = a_{n+d}$ for all $n > N$.

Solution. We will make repeated use of the following simple observation:

Lemma 1. If a positive integer d divides a_n and a_{n-m} for some m and $n > 2m$, it also divides a_{n-2m}. If d divides a_n and a_{n-2m}, it also divides a_{n-m}.

Proof. Both parts are obvious since a_n divides $a_{n-2m} + a_{n-m}$. □

Claim. The sequence (a_n) is bounded.

Proof. Suppose the contrary. Then there exist infinitely many indices n such that a_n is greater than each of the previous terms $a_1, a_2, \dots, a_{n-1}$. Let $a_n = k$ be such a term, $n > 10$. For each $s < \frac{n}{2}$ the number $a_n = k$ divides $a_{n-s} + a_{n-2s} < 2k$, therefore

$$a_{n-s} + a_{n-2s} = k.$$

In particular,

$$a_n = a_{n-1} + a_{n-2} = a_{n-2} + a_{n-4} = a_{n-4} + a_{n-8},$$

that is, $a_{n-1} = a_{n-4}$ and $a_{n-2} = a_{n-8}$. It follows from Lemma 1 that a_{n-1} divides a_{n-1-3s} for $3s < n-1$ and a_{n-2} divides a_{n-2-6s} for $6s < n-2$. Since at least one of the numbers a_{n-1} and a_{n-2} is at least $a_n/2$, so is some a_i with $i \leqslant 6$. However, a_n can be arbitrarily large, a contradiction. □

Since (a_n) is bounded, there exist only finitely many i for which a_i appears in the sequence finitely many times. In other words, there exists N such that if $a_i = t$ and $i > N$, then $a_j = t$ for infinitely many j.

Clearly the sequence $(a_{n+N})_{n>0}$ satisfies the divisibility condition, and it is enough to prove that this sequence is eventually periodic. Thus truncating the sequence if necessary, we can assume that each number appears infinitely many times in the sequence. Let k be the maximum number appearing in the sequence.

Lemma 2. If a positive integer d divides a_n for some n, then the numbers i such that d divides a_i form an arithmetical progression with an odd difference.

Proof. Let $i_1 < i_2 < i_3 < \dots$ be all the indices i such that d divides a_i. If $i_s + i_{s+1}$ is even, it follows from Lemma 1 that d also divides $a_{\frac{i_s+i_{s+1}}{2}}$, impossible since $i_s < \frac{i_s+i_{s+1}}{2} < i_{s+1}$. Thus i_s and i_{s+1} are always of different parity, and therefore $i_s + i_{s+2}$ is even. Applying Lemma 1 again, we see that d divides $a_{\frac{i_s+i_{s+2}}{2}}$, hence $\frac{i_s+i_{s+2}}{2} = i_{s+1}$, □

We are ready now to solve the problem.

The number of positive divisors of all terms of the progression is finite. Let d_s be the difference of the progression corresponding to s, that is, s divides a_n if and only if it divides a_{n+td_s} for any positive integer t. Let D be the product of all d_s. Then each s dividing a term of the progression divides a_n if and only if it divides a_{n+D}. This means that the sets of divisors of a_n and a_{n+D} coincide, and $a_{n+D} = a_n$. Thus D is a period of the sequence.

Comment. In the above solution we did not try to find the exact structure of the periodic part of (a_n). A little addition to the argument above shows that the period of the sequence has one of the following three forms:

(i) t (in this case the sequence is eventually constant);

(ii) t, $2t$, $3t$ or $2t$, t, $3t$ (so the period is 3);

(iii) $t, t, \dots, 2t$ (the period can be any odd number).

In these three cases t can be any positive integer. It is easy to see that all three cases satisfy the original condition.

We again denote by k be the maximum number appearing in the sequence. All the indices i such that $a_i = k$ form an arithmetical progression. If the difference of this progression is 1, the sequence (a_n) is constant, and we get the case (i). Assume that the difference T is at least 3.

Take an index n such that $a_n = k$ and let $a = a_{n-2}$, $b = a_{n-1}$. We have $a, b < k$ and therefore $k = a_n = a_{n-1} + a_{n-2} = a + b$. If $a = b = \frac{k}{2}$, then all the terms $a_1, a_2, \dots, a_n$ are divisible by $k/2$, that is, are equal to k or $k/2$. Since the indices i such that $a_i = k$ form an arithmetical progression with odd diference, we get the case (iii).

Suppose now that $a \neq b$.

Claim. For $\frac{n}{2} < m < n$ we have $a_m = a$ if $m \equiv n-2 \pmod 3$ and $a_m = b$ if $m \equiv n-1 \pmod 3$.

Proof. The number $k = a_n$ divides $a_{n-2} + a_{n-1} = a + b$ and $a_{n-4} + a_{n-2} = a_{n-4} + a$ and is therefore equal to these sums (since $a, b < k$ and $a_i \leqslant k$ for all i). Therefore $a_{n-1} = a_{n-4} = b$, that is, $a_{n-4} < k$, $a_{n-4} + a_{n-8} = k$ and $a_{n-8} = a_{n-2} = a$. One of the numbers a and b is greater than $k/2$.

If $b = a_{n-1} = a_{n-4} > \frac{k}{2}$, it follows from Lemma 1 that a_{n-1} divides a_{n-1-3s} when $3s < n-1$, and therefore $a_{n-1-3s} = b$ when $3s < n-1$. When $6s < n-4$, k also divides $a_{n-4-6s} + a_{n-2-3s} = b + a_{n-2-3s}$, thus, $a_{n-2-3s} = k - b = a$.

If $a = a_{n-2} = a_{n-8} > \frac{k}{2}$, all the terms a_{n-2-6s} with $6s < n-2$ are divisible by a, that is, the indices i for which a divides a_i form a progression with difference dividing 6. Since this difference is odd and greater than 1, it must be 3, that is, $a_{n-2-3s} = a$ when $3s < n-2$. Similarly to the previous case, we have $a_{n-1-3s} = a_n - a_{n-2-6s} = k - a = b$ when $6s < n-2$. □

Let a_n and a_{n+T} be two consecutive terms of the sequence equal to k. If n is large enough, $\frac{n+T}{2} < n-2$, and applying the claim to $n+T$ instead of n we see that the three consecutive terms $a_{n-2} = a$, $a_{n-1} = b$, $a_n = k$ must be equal to a_{n+T-2}, a_{n+T-1} and a_{n+T} respectively. Thus, for some i we have $a_{i+3s} = a$ and $a_{i+1+3s} = b$ for all s. Truncating the sequence again if necessary, we may assume that $a_{3s+1} = a$ and $a_{3s+2} = b$ for all s. We know also that $a_n = k$ if and only if n is divisible by T (incidentally, this proves that T is divisible by 3).

If $a_{3s} = c$ for some integer s, each of the numbers a, b, c divides the sum of the other two. It is easy to see that these numbers are proportional to one of the triplets (1, 1, 1), (1, 1, 2) and (1, 2, 3) in some order. It follows that the greater of the two numbers a and b is the smaller multiplied by 2, 3 or 3/2. The last two cases are impossible because then c cannot be the maximum element in the triplet (a, b, c), while $c = k = a + b$ for infinitely many s. Thus the only possible case is 2, the numbers a and b are $k/3$ and $2k/3$ in some order, and the only possible values of c are k and $k/3$. Suppose that $a_{3s} = k/3$ for some $s > 1$. We can choose s so that $a_{3s+3} = k$. Therefore T, which we already know to be odd and divisible by 3, is greater than 3, that is, at least 9. Then $a_{3s-3} \neq k$, and the only other possibility is $a_{3s-3} = k/3$. However, $a_{3s+3} = k$ must divide $a_{3s} + a_{3s-3} = 2k/3$, which is impossible. We have proved then that $a_{3s} = k$ for all $s > 1$, which is the case (ii).

N8. For a polynomial $P(x)$ with integer coefficients let $P^1(x) = P(x)$ and $P^{k+1}(x) = P(P^k(x))$ for $k \geqslant 1$. Find all positive integers n for which there exists a polynomial $P(x)$ with integer coefficients such that for every integer $m \geqslant 1$, the numbers $P^m(1), \ldots, P^m(n)$ leave exactly $\lceil n/2^m \rceil$ distinct remainders when divided by n.

Answer: All powers of 2 and all primes.

Solution. Denote the set of residues modulo ℓ by $\mathbb{Z}_\ell$. Observe that P can be regarded as a function $\mathbb{Z}_\ell \to \mathbb{Z}_\ell$ for any positive integer ℓ. Denote the cardinality of the set $P^m(\mathbb{Z}_\ell)$ by $f_{m,\ell}$. Note that $f_{m,n} = \lceil n/2^m \rceil$ for all $m \geqslant 1$ if and only if $f_{m+1,n} = \lceil f_{m,n}/2 \rceil$ for all $m \geqslant 0$.

Part 1. The required polynomial exists when n is a power of 2 or a prime.

If n is a power of 2, set $P(x) = 2x$.

If $n = p$ is an odd prime, every function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ coincides with some polynomial with integer coefficients. So we can pick the function that sends $x \in \{0, 1, \ldots, p-1\}$ to $\lfloor x/2 \rfloor$.

Part 2. The required polynomial does not exist when n is not a prime power.

Let $n = ab$ where $a, b > 1$ and $\gcd(a, b) = 1$. Note that, since $\gcd(a, b) = 1$,

$$f_{m,ab} = f_{m,a} f_{m,b}$$

by the Chinese remainder theorem. Also, note that, if $f_{m,\ell} = f_{m+1,\ell}$, then P permutes the image of P^m on $\mathbb{Z}_\ell$, and therefore $f_{s,\ell} = f_{m,\ell}$ for all $s > m$. So, as $f_{m,ab} = 1$ for sufficiently large m, we have for each m

$$f_{m,a} > f_{m+1,a} \quad \text{or} \quad f_{m,a} = 1, \qquad f_{m,b} > f_{m+1,b} \quad \text{or} \quad f_{m,b} = 1.$$

Choose the smallest m such that $f_{m+1,a} = 1$ or $f_{m+1,b} = 1$. Without loss of generality assume that $f_{m+1,a} = 1$. Then $f_{m+1,ab} = f_{m+1,b} < f_{m,b} \leqslant f_{m,ab}/2 \leqslant f_{m+1,ab}$, a contradiction.

Part 3. The required polynomial does not exist when n is an odd prime power that is not a prime.

Let $n = p^k$, where $p \geqslant 3$ is prime and $k \geqslant 2$. For $r \in \mathbb{Z}_p$ let S_r denote the subset of $\mathbb{Z}_{p^k}$ consisting of numbers congruent to r modulo p. We denote the cardinality of a set S by $|S|$.

Claim. For any residue r modulo p, either $|P(S_r)| = p^{k-1}$ or $|P(S_r)| \leqslant p^{k-2}$.

Proof. Recall that $P(r+h) = P(r) + hP'(r) + h^2 Q(r, h)$, where Q is an integer polynomial.

If $p \mid P'(r)$, then $P(r + ps) \equiv P(r) \pmod{p^2}$, hence all elements of $P(S_r)$ are congruent modulo p^2. So in this case $|P(S_r)| \leqslant p^{k-2}$.

Now we show that $p \nmid P'(r)$ implies $|P(S_r)| = p^{k-1}$ for all k.

Suppose the contrary: $|P(S_r)| < p^{k-1}$ for some $k > 1$. Let us choose the smallest k for which this is so. To each residue in $P(S_r)$ we assign its residue modulo p^{k-1}; denote the resulting set by $\overline{P}(S, r)$. We have $|\overline{P}(S, r)| = p^{k-2}$ by virtue of minimality of k. Then $|P(S_r)| < p^{k-1} = p \cdot |\overline{P}(S, r)|$, that is, there is $u = P(x) \in P(S_r)$ ($x \equiv r \pmod{p}$) and $t \not\equiv 0 \pmod{p}$ such that $u + p^{k-1}t \notin P(S_r)$.

Note that $P(x + p^{k-1}s) \equiv u + p^{k-1}sP'(x) \pmod{p^k}$. Since $P(x + p^{k-1}s) \not\equiv u + p^{k-1}t \pmod{p^k}$, the congruence $p^{k-1}sP'(x) \equiv p^{k-1}t \pmod{p^k}$ has no solutions. So the congruence $sP'(x) \equiv t \pmod{p}$ has no solutions, which contradicts $p \nmid P'(r)$. $\square$

Since the image of P^m consists of one element for sufficiently large m, we can take the smallest m such that $|P^{m-1}(S_r)| = p^{k-1}$ for some $r \in \mathbb{Z}_p$, but $|P^m(S_q)| \leqslant p^{k-2}$ for all $q \in \mathbb{Z}_p$.

From now on, we fix m and r.

Since the image of $P^{m-1}(\mathbb{Z}_{p^k}) \backslash P^{m-1}(S_r)$ under P contains $P^m(\mathbb{Z}_{p^k}) \backslash P^m(S_r)$, we have

$$a := |P^m(\mathbb{Z}_{p^k}) \backslash P^m(S_r)| \leqslant |P^{m-1}(\mathbb{Z}_{p^k}) \backslash P^{m-1}(S_r)|,$$

thus

$$a + p^{k-1} \leqslant f_{m-1,p^k} \leqslant 2f_{m,p^k} \leqslant 2p^{k-2} + 2a,$$

so

$$(p-2)p^{k-2} \leqslant a.$$

Since $f_{i,p} = 1$ for sufficiently large i, there is exactly one $t \in \mathbb{Z}_p$ such that $P(t) \equiv t \pmod{p}$. Moreover, as i increases, the cardinality of the set $\{s \in \mathbb{Z}_p \mid P^i(s) \equiv t \pmod{p}\}$ increases (strictly), until it reaches the value p. So either

$$|\{s \in \mathbb{Z}_p \mid P^{m-1}(s) \equiv t \pmod{p}\}| = p \quad \text{or} \quad |\{s \in \mathbb{Z}_p \mid P^{m-1}(s) \equiv t \pmod{p}\}| \geqslant m.$$

Therefore, either $f_{m-1,p} = 1$ or there exists a subset $X \subset \mathbb{Z}_p$ of cardinality at least m such that $P^{m-1}(x) \equiv t \pmod{p}$ for all $x \in X$.

In the first case $|P^{m-1}(\mathbb{Z}_{p^k})| \leqslant p^{k-1} = |P^{m-1}(S_r)|$, so $a = 0$, a contradiction.

In the second case let Y be the set of all elements of $\mathbb{Z}_{p^k}$ congruent to some element of X modulo p. Let $Z = \mathbb{Z}_{p^k} \backslash Y$. Then $P^{m-1}(Y) \subset S_t$, $P(S_t) \subsetneq S_t$, and $Z = \bigcup_{i \in \mathbb{Z}_p \backslash X} S_i$, so

$$|P^m(Y)| \leqslant |P(S_t)| \leqslant p^{k-2} \quad \text{and} \quad |P^m(Z)| \leqslant |\mathbb{Z}_p \backslash X| \cdot p^{k-2} \leqslant (p-m)p^{k-2}.$$

Hence,

$$(p-2)p^{k-2} \leqslant a < |P^m(\mathbb{Z}_{p^k})| \leqslant |P^m(Y)| + |P^m(Z)| \leqslant (p-m+1)p^{k-2}$$

and $m < 3$. Then $|P^2(S_q)| \leqslant p^{k-2}$ for all $q \in \mathbb{Z}_p$, so

$$p^k/4 \leqslant |P^2(\mathbb{Z}_{p^k})| \leqslant p^{k-1},$$

which is impossible for $p \geqslant 5$. It remains to consider the case $p = 3$.

As before, let t be the only residue modulo 3 such that $P(t) \equiv t \pmod{3}$.

If $3 \nmid P'(t)$, then $P(S_t) = S_t$ by the proof of the Claim above, which is impossible.

So $3 \mid P'(t)$. By substituting $h = 3^i s$ into the formula $P(t+h) = P(t) + hP'(t) + h^2 Q(t,h)$, we obtain $P(t + 3^i s) \equiv P(t) \pmod{3^{i+1}}$. Using induction on i we see that all elements of $P^i(S_t)$ are congruent modulo 3^{i+1}. Thus, $|P^{k-1}(S_t)| = 1$.

Note that $f_{1,3} \leqslant 2$ and $f_{2,3} \leqslant 1$, so $P^2(\mathbb{Z}_{3^k}) \subset S_t$. Therefore, $|P^{k+1}(\mathbb{Z}_{3^k})| \leqslant |P^{k-1}(S_t)| = 1$. It follows that $3^k \leqslant 2^{k+1}$, which is impossible for $k \geqslant 2$.

Comment. Here is an alternative version of the problem.

A function $f\colon \mathbb{Z} \to \mathbb{Z}$ is chosen so that $a - b \mid f(a) - f(b)$ for all $a, b \in \mathbb{Z}$ with $a \neq b$. Let $S_0 = \mathbb{Z}$, and for each positive integer m, let S_m denote the image of f on the set S_{m-1}. It is given that, for each nonnegative integer m, there are exactly $\lceil n/2^m \rceil$ distinct residues modulo n in the set S_m. Find all possible values of n.

Answer: All powers of primes.

Solution. Observe that f can be regarded as a function $\mathbb{Z}_\ell \to \mathbb{Z}_\ell$ for any positive integer ℓ. We use notations f^m and $f_{m,\ell}$ as in the above solution.

Part 1. There exists a function $f\colon \mathbb{Z}_{p^k} \to \mathbb{Z}_{p^k}$ satisfying the desired properties.

For $x \in \mathbb{Z}_{p^k}$, let $\operatorname{rev}(x)$ denote the reversal of the base-p digits of x (we write every $x \in \mathbb{Z}_{p^k}$ with exactly k digits, adding zeroes at the beginning if necessary). Choose

$$f(x) = \operatorname{rev}\left(\left\lfloor \frac{\operatorname{rev}(x)}{2} \right\rfloor\right)$$

where, for dividing by 2, $\operatorname{rev}(x)$ is interpreted as an integer in the range $[0, p^k)$. It is easy to see that $f_{m+1,k} = \lceil f_{m,k}/2 \rceil$.

We claim that if $a, b \in \mathbb{Z}_{p^k}$ so that $p^m | a - b$, then $p^m | f(a) - f(b)$. Let $x = \operatorname{rev}(a)$, $y = \operatorname{rev}(b)$. The first m digits of x and y are the same, i.e $\lfloor x/p^{m-k} \rfloor = \lfloor y/p^{m-k} \rfloor$. For every positive integers c, d and z we have $\lfloor \lfloor z/c \rfloor / d \rfloor = \lfloor z/(cd) \rfloor = \lfloor \lfloor z/d \rfloor / c \rfloor$, so

$$\left\lfloor \lfloor x/2 \rfloor / p^{m-k} \right\rfloor = \left\lfloor \lfloor x/p^{m-k} \rfloor / 2 \right\rfloor = \left\lfloor \lfloor y/p^{m-k} \rfloor / 2 \right\rfloor = \left\lfloor \lfloor y/2 \rfloor / p^{m-k} \right\rfloor.$$

Thus, the first m digits of $\lfloor x/2 \rfloor$ and $\lfloor y/2 \rfloor$ are the same. So the last m digits of $f(a)$ and $f(b)$ are the same, i.e. $p^m | f(a) - f(b)$.

Part 2. Lifting the function $f\colon \mathbb{Z}_{p^k} \to \mathbb{Z}_{p^k}$ to a function on all of $\mathbb{Z}$.

We show that, for any function $f\colon \mathbb{Z}_{p^k} \to \mathbb{Z}_{p^k}$ for which $\gcd(p^k, a-b) \mid f(a) - f(b)$, there is a corresponding function $g\colon \mathbb{Z} \to \mathbb{Z}$ for which $a - b \mid g(a) - g(b)$ for all distinct integers a, b and $g(x) \equiv f(x) \pmod{p^k}$ for all $x \in \mathbb{Z}$, whence the proof will be completed. We will construct the values of such a function inductively; assume that we have constructed it for some interval $[a, b)$ and wish to define $g(b)$. (We will define $g(a-1)$ similarly.)

For every prime $q \leqslant |a - b|$, we choose the maximal α_q for which there exists $c_q \in [a, b)$, such that $b - c_q \,\vdots\, q^{\alpha_q}$, and choose one such c_q.

We apply Chinese remainder theorem to find $g(b)$ satisfying the following conditions:

$$g(b) \equiv g(c_q) \pmod{q^{\alpha_q}} \quad \text{for} \quad q \neq p, \quad \text{and}$$
$$g(b) \equiv g(c_p) \pmod{q^{\alpha_p}} \quad \text{if} \quad \alpha_p \geqslant k, \qquad g(b) \equiv f(b) \pmod{p^k} \quad \text{if} \quad \alpha_p < k.$$

It is not hard to verify that $b - c \mid g(b) - g(c)$ for every $c \in [a, b)$ and $g(b) \equiv f(b) \pmod{p^k}$.

Part 3. The required function does not exist if n has at least two different prime divisors.

The proof is identical to the polynomial version.