IMO 2020

61st International
Mathematical
Olympiad
Saint Petersburg
Russia

# SHORTLISTED PROBLEMS

(with solutions)

61st International Mathematical Olympiad

Saint-Petersburg — Russia, 18–28 September 2020

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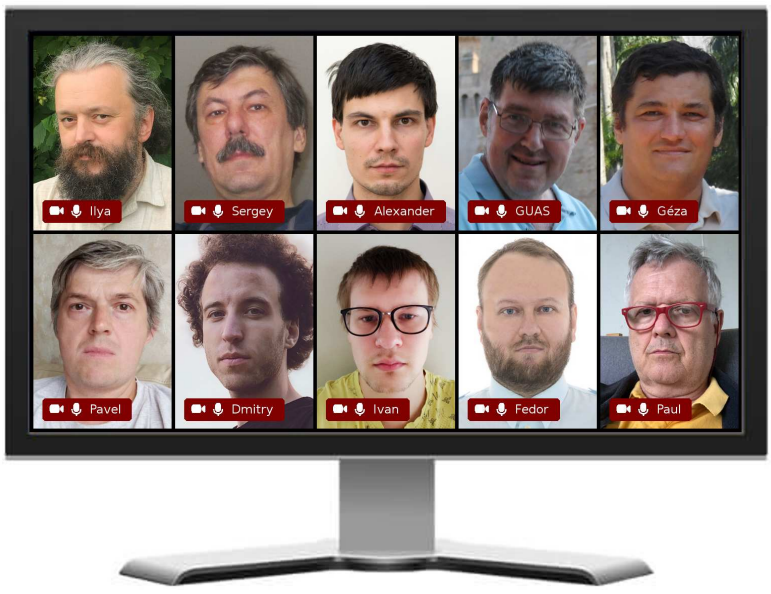
## Problem Selection Committee

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# Problems

## Algebra

**A1.** *Version 1.* Let $n$ be a positive integer, and set $N = 2^n$. Determine the smallest real number $a_n$ such that, for all real $x$,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant a_n(x-1)^2 + x.$$

*Version 2.* For every positive integer $N$, determine the smallest real number $b_N$ such that, for all real $x$,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant b_N(x-1)^2 + x.$$

*(Ireland)*

**A2.** Let $\mathcal{A}$ denote the set of all polynomials in three variables $x, y, z$ with integer coefficients. Let $\mathcal{B}$ denote the subset of $\mathcal{A}$ formed by all polynomials which can be expressed as

$$(x+y+z)P(x,y,z) + (xy+yz+zx)Q(x,y,z) + xyzR(x,y,z)$$

with $P, Q, R \in \mathcal{A}$. Find the smallest non-negative integer $n$ such that $x^i y^j z^k \in \mathcal{B}$ for all non-negative integers $i, j, k$ satisfying $i + j + k \geqslant n$.

*(Venezuela)*

**A3.** Suppose that $a$, $b$, $c$, $d$ are positive real numbers satisfying $(a+c)(b+d) = ac + bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

*(Israel)*

**A4.** Let $a, b, c, d$ be four real numbers such that $a \geqslant b \geqslant c \geqslant d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)\, a^a\, b^b\, c^c\, d^d < 1.$$

*(Belgium)*

**A5.** A magician intends to perform the following trick. She announces a positive integer $n$, along with $2n$ real numbers $x_1 < \ldots < x_{2n}$, to the audience. A member of the audience then secretly chooses a polynomial $P(x)$ of degree $n$ with real coefficients, computes the $2n$ values $P(x_1), \ldots, P(x_{2n})$, and writes down these $2n$ values on the blackboard in non-decreasing order. After that the magician announces the secret polynomial to the audience.

Can the magician find a strategy to perform such a trick?

*(Luxembourg)*

**A6.** Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$f^{a^2+b^2}(a+b) = af(a) + bf(b) \qquad \text{for every } a, b \in \mathbb{Z}.$$

Here, $f^n$ denotes the $n^{\text{th}}$ iteration of $f$, i.e., $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for all $n \geqslant 0$.

*(Slovakia)*

**A7.** Let $n$ and $k$ be positive integers. Prove that for $a_1, \ldots, a_n \in [1, 2^k]$ one has

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{a_1^2 + \ldots + a_i^2}} \leqslant 4\sqrt{kn}.$$

*(Iran)*

**A8.** Let $\mathbb{R}^+$ be the set of positive real numbers. Determine all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all positive real numbers $x$ and $y$,

$$f\big(x + f(xy)\big) + y = f(x)f(y) + 1.$$

*(Ukraine)*

# Combinatorics

**C1.** Let $n$ be a positive integer. Find the number of permutations $a_1, a_2, \ldots, a_n$ of the sequence $1, 2, \ldots, n$ satisfying

$$a_1 \leqslant 2a_2 \leqslant 3a_3 \leqslant \ldots \leqslant na_n.$$

*(United Kingdom)*

**C2.** In a regular 100-gon, 41 vertices are colored black and the remaining 59 vertices are colored white. Prove that there exist 24 convex quadrilaterals $Q_1, \ldots, Q_{24}$ whose corners are vertices of the 100-gon, so that

- the quadrilaterals $Q_1, \ldots, Q_{24}$ are pairwise disjoint, and
- every quadrilateral $Q_i$ has three corners of one color and one corner of the other color.

*(Austria)*

**C3.** Let $n$ be an integer with $n \geqslant 2$. On a slope of a mountain, $n^2$ checkpoints are marked, numbered from 1 to $n^2$ from the bottom to the top. Each of two cable car companies, $A$ and $B$, operates $k$ cable cars numbered from 1 to $k$; each cable car provides a transfer from some checkpoint to a higher one. For each company, and for any $i$ and $j$ with $1 \leqslant i < j \leqslant k$, the starting point of car $j$ is higher than the starting point of car $i$; similarly, the finishing point of car $j$ is higher than the finishing point of car $i$. Say that two checkpoints are *linked* by some company if one can start from the lower checkpoint and reach the higher one by using one or more cars of that company (no movement on foot is allowed).

Determine the smallest $k$ for which one can guarantee that there are two checkpoints that are linked by each of the two companies.

*(India)*

**C4.** The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 1$. Given an integer $n \geqslant 2$, determine the smallest size of a set $S$ of integers such that for every $k = 2, 3, \ldots, n$ there exist some $x, y \in S$ such that $x - y = F_k$.

*(Croatia)*

**C5.** Let $p$ be an odd prime, and put $N = \frac{1}{4}(p^3 - p) - 1$. The numbers $1, 2, \ldots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leqslant N$, denote by $r(n)$ the fraction of integers in $\{1, 2, \ldots, n\}$ that are red.

Prove that there exists a positive integer $a \in \{1, 2, \ldots, p-1\}$ such that $r(n) \neq a/p$ for all $n = 1, 2, \ldots, N$.

*(Netherlands)*

**C6.** $4n$ coins of weights $1, 2, 3, \ldots, 4n$ are given. Each coin is colored in one of $n$ colors and there are four coins of each color. Show that all these coins can be partitioned into two sets with the same total weight, such that each set contains two coins of each color.

*(Hungary)*

**C7.** Consider any rectangular table having finitely many rows and columns, with a real number $a(r, c)$ in the cell in row $r$ and column $c$. A pair $(R, C)$, where $R$ is a set of rows and $C$ a set of columns, is called a *saddle pair* if the following two conditions are satisfied:

(*i*) For each row $r'$, there is $r \in R$ such that $a(r, c) \geqslant a(r', c)$ for all $c \in C$;

(*ii*) For each column $c'$, there is $c \in C$ such that $a(r, c) \leqslant a(r, c')$ for all $r \in R$.

A saddle pair $(R, C)$ is called a *minimal pair* if for each saddle pair $(R', C')$ with $R' \subseteq R$ and $C' \subseteq C$, we have $R' = R$ and $C' = C$.

Prove that any two minimal pairs contain the same number of rows.

*(Thailand)*

**C8.** Players $A$ and $B$ play a game on a blackboard that initially contains 2020 copies of the number 1. In every round, player $A$ erases two numbers $x$ and $y$ from the blackboard, and then player $B$ writes one of the numbers $x + y$ and $|x - y|$ on the blackboard. The game terminates as soon as, at the end of some round, one of the following holds:

(1) one of the numbers on the blackboard is larger than the sum of all other numbers;

(2) there are only zeros on the blackboard.

Player $B$ must then give as many cookies to player $A$ as there are numbers on the blackboard. Player $A$ wants to get as many cookies as possible, whereas player $B$ wants to give as few as possible. Determine the number of cookies that $A$ receives if both players play optimally.

*(Austria)*

# Geometry

**G1.** Let $ABC$ be an isosceles triangle with $BC = CA$, and let $D$ be a point inside side $AB$ such that $AD < DB$. Let $P$ and $Q$ be two points inside sides $BC$ and $CA$, respectively, such that $\angle DPB = \angle DQA = 90^\circ$. Let the perpendicular bisector of $PQ$ meet line segment $CQ$ at $E$, and let the circumcircles of triangles $ABC$ and $CPQ$ meet again at point $F$, different from $C$.

Suppose that $P, E, F$ are collinear. Prove that $\angle ACB = 90^\circ$.

*(Luxembourg)*

**G2.** Let $ABCD$ be a convex quadrilateral. Suppose that $P$ is a point in the interior of $ABCD$ such that

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

The internal bisectors of angles $ADP$ and $PCB$ meet at a point $Q$ inside the triangle $ABP$. Prove that $AQ = BQ$.

*(Poland)*

**G3.** Let $ABCD$ be a convex quadrilateral with $\angle ABC > 90^\circ$, $\angle CDA > 90^\circ$, and $\angle DAB = \angle BCD$. Denote by $E$ and $F$ the reflections of $A$ in lines $BC$ and $CD$, respectively. Suppose that the segments $AE$ and $AF$ meet the line $BD$ at $K$ and $L$, respectively. Prove that the circumcircles of triangles $BEK$ and $DFL$ are tangent to each other.

*(Slovakia)*

**G4.** In the plane, there are $n \geqslant 6$ pairwise disjoint disks $D_1, D_2, \ldots, D_n$ with radii $R_1 \geqslant R_2 \geqslant \ldots \geqslant R_n$. For every $i = 1, 2, \ldots, n$, a point $P_i$ is chosen in disk $D_i$. Let $O$ be an arbitrary point in the plane. Prove that

$$OP_1 + OP_2 + \ldots + OP_n \geqslant R_6 + R_7 + \ldots + R_n.$$

(A disk is assumed to contain its boundary.)

*(Iran)*

**G5.** Let $ABCD$ be a cyclic quadrilateral with no two sides parallel. Let $K$, $L$, $M$, and $N$ be points lying on sides $AB$, $BC$, $CD$, and $DA$, respectively, such that $KLMN$ is a rhombus with $KL \parallel AC$ and $LM \parallel BD$. Let $\omega_1$, $\omega_2$, $\omega_3$, and $\omega_4$ be the incircles of triangles $ANK$, $BKL$, $CLM$, and $DMN$, respectively. Prove that the internal common tangents to $\omega_1$ and $\omega_3$ and the internal common tangents to $\omega_2$ and $\omega_4$ are concurrent.

*(Poland)*

**G6.** Let $I$ and $I_A$ be the incenter and the $A$-excenter of an acute-angled triangle $ABC$ with $AB < AC$. Let the incircle meet $BC$ at $D$. The line $AD$ meets $BI_A$ and $CI_A$ at $E$ and $F$, respectively. Prove that the circumcircles of triangles $AID$ and $I_AEF$ are tangent to each other.

*(Slovakia)*

**G7.** Let $P$ be a point on the circumcircle of an acute-angled triangle $ABC$. Let $D$, $E$, and $F$ be the reflections of $P$ in the midlines of triangle $ABC$ parallel to $BC$, $CA$, and $AB$, respectively. Denote by $\omega_A$, $\omega_B$, and $\omega_C$ the circumcircles of triangles $ADP$, $BEP$, and $CFP$, respectively. Denote by $\omega$ the circumcircle of the triangle formed by the perpendicular bisectors of segments $AD$, $BE$ and $CF$.

Show that $\omega_A$, $\omega_B$, $\omega_C$, and $\omega$ have a common point.

*(Denmark)*

**G8.** Let $\Gamma$ and $I$ be the circumcircle and the incenter of an acute-angled triangle $ABC$. Two circles $\omega_B$ and $\omega_C$ passing through $B$ and $C$, respectively, are tangent at $I$. Let $\omega_B$ meet the shorter arc $AB$ of $\Gamma$ and segment $AB$ again at $P$ and $M$, respectively. Similarly, let $\omega_C$ meet the shorter arc $AC$ of $\Gamma$ and segment $AC$ again at $Q$ and $N$, respectively. The rays $PM$ and $QN$ meet at $X$, and the tangents to $\omega_B$ and $\omega_C$ at $B$ and $C$, respectively, meet at $Y$.

Prove that the points $A$, $X$, and $Y$ are collinear.

*(Netherlands)*

**G9.** Prove that there exists a positive constant $c$ such that the following statement is true:

Assume that $n$ is an integer with $n \geqslant 2$, and let $\mathcal{S}$ be a set of $n$ points in the plane such that the distance between any two distinct points in $\mathcal{S}$ is at least 1. Then there is a line $\ell$ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to $\ell$ is at least $cn^{-1/3}$.

(A line $\ell$ *separates* a point set $\mathcal{S}$ if some segment joining two points in $\mathcal{S}$ crosses $\ell$.)

*(Taiwan)*

# Number Theory

**N1.** Given a positive integer $k$, show that there exists a prime $p$ such that one can choose distinct integers $a_1, a_2, \ldots, a_{k+3} \in \{1, 2, \ldots, p-1\}$ such that $p$ divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, \ldots, k$.

*(South Africa)*

**N2.** For each prime $p$, there is a kingdom of $p$-Landia consisting of $p$ islands numbered 1, 2, ..., $p$. Two distinct islands numbered $n$ and $m$ are connected by a bridge if and only if $p$ divides $(n^2 - m + 1)(m^2 - n + 1)$. The bridges may pass over each other, but cannot cross. Prove that for infinitely many $p$ there are two islands in $p$-Landia not connected by a chain of bridges.

*(Denmark)*

**N3.** Let $n$ be an integer with $n \geqslant 2$. Does there exist a sequence $(a_1, \ldots, a_n)$ of positive integers with not all terms being equal such that the arithmetic mean of every two terms is equal to the geometric mean of some (one or more) terms in this sequence?

*(Estonia)*

**N4.** For any odd prime $p$ and any integer $n$, let $d_p(n) \in \{0, 1, \ldots, p-1\}$ denote the remainder when $n$ is divided by $p$. We say that $(a_0, a_1, a_2, \ldots)$ is a *$p$-sequence*, if $a_0$ is a positive integer coprime to $p$, and $a_{n+1} = a_n + d_p(a_n)$ for $n \geqslant 0$.

(a) Do there exist infinitely many primes $p$ for which there exist $p$-sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ such that $a_n > b_n$ for infinitely many $n$, and $b_n > a_n$ for infinitely many $n$?

(b) Do there exist infinitely many primes $p$ for which there exist $p$-sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ such that $a_0 < b_0$, but $a_n > b_n$ for all $n \geqslant 1$?

*(United Kingdom)*

**N5.** Determine all functions $f$ defined on the set of all positive integers and taking non-negative integer values, satisfying the three conditions:

*(i)* $f(n) \neq 0$ for at least one $n$;

*(ii)* $f(xy) = f(x) + f(y)$ for every positive integers $x$ and $y$;

*(iii)* there are infinitely many positive integers $n$ such that $f(k) = f(n-k)$ for all $k < n$.

*(Croatia)*

**N6.** For a positive integer $n$, let $d(n)$ be the number of positive divisors of $n$, and let $\varphi(n)$ be the number of positive integers not exceeding $n$ which are coprime to $n$. Does there exist a constant $C$ such that

$$\frac{\varphi(d(n))}{d(\varphi(n))} \leqslant C$$

for all $n \geqslant 1$?

*(Cyprus)*

**N7.** Let $\mathcal{S}$ be a set consisting of $n \geqslant 3$ positive integers, none of which is a sum of two other distinct members of $\mathcal{S}$. Prove that the elements of $\mathcal{S}$ may be ordered as $a_1, a_2, \ldots, a_n$ so that $a_i$ does not divide $a_{i-1} + a_{i+1}$ for all $i = 2, 3, \ldots, n-1$.

*(Ukraine)*

# Solutions

## Algebra

**A1.** *Version 1.* Let $n$ be a positive integer, and set $N = 2^n$. Determine the smallest real number $a_n$ such that, for all real $x$,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant a_n(x-1)^2 + x.$$

*Version 2.* For every positive integer $N$, determine the smallest real number $b_N$ such that, for all real $x$,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant b_N(x-1)^2 + x.$$

*(Ireland)*

**Answer for both versions :** $a_n = b_N = N/2$.

**Solution 1 (for Version 1).** First of all, assume that $a_n < N/2$ satisfies the condition. Take $x = 1 + t$ for $t > 0$, we should have

$$\frac{(1+t)^{2N}+1}{2} \leqslant (1 + t + a_n t^2)^N.$$

Expanding the brackets we get

$$(1 + t + a_n t^2)^N - \frac{(1+t)^{2N}+1}{2} = \left(Na_n - \frac{N^2}{2}\right) t^2 + c_3 t^3 + \ldots + c_{2N} t^{2N} \tag{1}$$

with some coefficients $c_3, \ldots, c_{2N}$. Since $a_n < N/2$, the right hand side of (1) is negative for sufficiently small $t$. A contradiction.

It remains to prove the following inequality

$$\sqrt[N]{\frac{1+x^{2N}}{2}} \leqslant x + \frac{N}{2}(x-1)^2, \tag{$\mathcal{I}(N,x)$}$$

where $N = 2^n$.

Use induction in $n$. The base case $n = 0$ is trivial: $N = 1$ and both sides of $\mathcal{I}(N,x)$ are equal to $(1 + x^2)/2$. For completing the induction we prove $\mathcal{I}(2N,x)$ assuming that $\mathcal{I}(N,y)$ is established for all real $y$. We have

$$\begin{aligned}\left(x + N(x-1)^2\right)^2 &= x^2 + N^2(x-1)^4 + N(x-1)^2\frac{(x+1)^2-(x-1)^2}{2} \\ &= x^2 + \frac{N}{2}(x^2-1)^2 + \left(N^2 - \frac{N}{2}\right)(x-1)^4 \geqslant x^2 + \frac{N}{2}(x^2-1)^2 \geqslant \sqrt[N]{\frac{1+x^{4N}}{2}},\end{aligned}$$

where the last inequality is $\mathcal{I}(N, x^2)$. Since

$$x + N(x-1)^2 \geqslant x + \frac{(x-1)^2}{2} = \frac{x^2+1}{2} \geqslant 0,$$

taking square root we get $\mathcal{I}(2N,x)$. The inductive step is complete.

**Solution 2.1 (for Version 2).** Like in Solution 1 of Version 1, we conclude that $b_N \geqslant N/2$. It remains to prove the inequality $\mathcal{I}(N,x)$ for an arbitrary positive integer $N$.

First of all, $\mathcal{I}(N,0)$ is obvious. Further, if $x>0$, then the left hand sides of $\mathcal{I}(N,-x)$ and $\mathcal{I}(N,x)$ coincide, while the right hand side of $\mathcal{I}(N,-x)$ is larger than that of $\mathcal{I}(N,-x)$ (their difference equals $2(N-1)x \geqslant 0$). Therefore, $\mathcal{I}(N,-x)$ follows from $\mathcal{I}(N,x)$. So, hereafter we suppose that $x>0$.

Divide $\mathcal{I}(N,x)$ by $x$ and let $t=(x-1)^2/x = x-2+1/x$; then $\mathcal{I}(n,x)$ reads as

$$f_N := \frac{x^N + x^{-N}}{2} \leqslant \left(1+\frac{N}{2}t\right)^N. \tag{2}$$

The key identity is the expansion of $f_N$ as a polynomial in $t$:

*Lemma.*

$$f_N = N\sum_{k=0}^{N} \frac{1}{N+k}\binom{N+k}{2k}t^k. \tag{3}$$

*Proof.* Apply induction on $N$. We will make use of the straightforward recurrence relation

$$f_{N+1} + f_{N-1} = (x+1/x)f_N = (2+t)f_N. \tag{4}$$

The base cases $N=1,2$ are straightforward:

$$f_1 = 1+\frac{t}{2}, \quad f_2 = \frac{1}{2}t^2+2t+1.$$

For the induction step from $N-1$ and $N$ to $N+1$, we compute the coefficient of $t^k$ in $f_{N+1}$ using the formula $f_{N+1} = (2+t)f_N - f_{N-1}$. For $k=0$ that coefficient equals 1, for $k>0$ it equals

$$\begin{aligned}
&2\frac{N}{N+k}\binom{N+k}{2k} + \frac{N}{N+k-1}\binom{N+k-1}{2k-2} - \frac{N-1}{N+k-1}\binom{N+k-1}{2k} \\
&= \frac{(N+k-1)!}{(2k)!(N-k)!}\left(2N + \frac{2k(2k-1)N}{(N+k-1)(N-k+1)} - \frac{(N-1)(N-k)}{N+k-1}\right) \\
&= \frac{(N+k-1)!}{(2k)!(N-k+1)!}\left(2N(N-k+1)+3kN+k-N^2-N\right) = \frac{\binom{N+k+1}{2k}}{(N+k+1)}(N+1),
\end{aligned}$$

that completes the induction. □

Turning back to the problem, in order to prove (2) we write

$$\left(1+\frac{N}{2}t\right)^N - f_N = \left(1+\frac{N}{2}t\right)^N - N\sum_{k=0}^{N}\frac{1}{N+k}\binom{N+k}{2k}t^k = \sum_{k=0}^{N}\alpha_k t^k,$$

where

$$\begin{aligned}
\alpha_k &= \left(\frac{N}{2}\right)^k\binom{N}{k} - \frac{N}{N+k}\binom{N+k}{2k} \\
&= \left(\frac{N}{2}\right)^k\binom{N}{k}\left(1 - 2^k\frac{(1+1/N)(1+2/N)\cdot\ldots\cdot(1+(k-1)/N)}{(k+1)\cdot\ldots\cdot(2k)}\right) \\
&\geqslant \left(\frac{N}{2}\right)^k\binom{N}{k}\left(1-2^k\frac{2\cdot3\cdot\ldots\cdot k}{(k+1)\cdot\ldots\cdot(2k)}\right) = \left(\frac{N}{2}\right)^k\binom{N}{k}\left(1-\prod_{j=1}^{k}\frac{2j}{k+j}\right) \geqslant 0,
\end{aligned}$$

and (2) follows.

**Solution 2.2 (for Version 2).** Here we present another proof of the inequality (2) for $x > 0$, or, equivalently, for $t = (x-1)^2/x \geqslant 0$. Instead of finding the coefficients of the polynomial $f_N = f_N(t)$ we may find its roots, which is in a sense more straightforward. Note that the recurrence (4) and the initial conditions $f_0 = 1$, $f_1 = 1 + t/2$ imply that $f_N$ is a polynomial in $t$ of degree $N$. It also follows by induction that $f_N(0) = 1, f_N'(0) = N^2/2$: the recurrence relations read as $f_{N+1}(0) + f_{N-1}(0) = 2f_N(0)$ and $f_{N+1}'(0) + f_{N-1}'(0) = 2f_N'(0) + f_N(0)$, respectively.

Next, if $x_k = \exp(\frac{i\pi(2k-1)}{2N})$ for $k \in \{1, 2, \ldots, N\}$, then

$$-t_k := 2 - x_k - \frac{1}{x_k} = 2 - 2\cos\frac{\pi(2k-1)}{2N} = 4\sin^2\frac{\pi(2k-1)}{4N} > 0$$

and

$$f_N(t_k) = \frac{x_k^N + x_k^{-N}}{2} = \frac{\exp\left(\frac{i\pi(2k-1)}{2}\right) + \exp\left(-\frac{i\pi(2k-1)}{2}\right)}{2} = 0.$$

So the roots of $f_N$ are $t_1, \ldots, t_N$ and by the AM–GM inequality we have

$$f_N(t) = \left(1 - \frac{t}{t_1}\right)\left(1 - \frac{t}{t_2}\right)\ldots\left(1 - \frac{t}{t_N}\right) \leqslant \left(1 - \frac{t}{N}\left(\frac{1}{t_1} + \ldots + \frac{1}{t_n}\right)\right)^N = \left(1 + \frac{tf_N'(0)}{N}\right)^N = \left(1 + \frac{N}{2}t\right)^N.$$

**Comment.** The polynomial $f_N(t)$ equals to $\frac{1}{2}T_N(t+2)$, where $T_n$ is the $n^{\text{th}}$ Chebyshev polynomial of the first kind: $T_n(2\cos s) = 2\cos ns$, $T_n(x + 1/x) = x^n + 1/x^n$.

**Solution 2.3 (for Version 2).** Here we solve the problem when $N \geqslant 1$ is an arbitrary real number. For a real number $a$ let

$$f(x) = \left(\frac{x^{2N}+1}{2}\right)^{\frac{1}{N}} - a(x-1)^2 - x.$$

Then $f(1) = 0$,

$$f'(x) = \left(\frac{x^{2N}+1}{2}\right)^{\frac{1}{N}-1} x^{2N-1} - 2a(x-1) - 1 \quad \text{and} \quad f'(1) = 0;$$

$$f''(x) = (1-N)\left(\frac{x^{2N}+1}{2}\right)^{\frac{1}{N}-2} x^{4N-2} + (2N-1)\left(\frac{x^{2N}+1}{2}\right)^{\frac{1}{N}-1} x^{2N-2} - 2a \quad \text{and} \quad f''(1) = N - 2a.$$

So if $a < \frac{N}{2}$, the function $f$ has a strict local minimum at point 1, and the inequality $f(x) \leqslant 0 = f(1)$ does not hold. This proves $b_N \geqslant N/2$.

For $a = \frac{N}{2}$ we have $f''(1) = 0$ and

$$f'''(x) = \frac{1}{2}(1-N)(1-2N)\left(\tfrac{x^{2N}+1}{2}\right)^{\frac{1}{N}-3} x^{2N-3}(1-x^{2N}) \quad \begin{cases} > 0 & \text{if } 0 < x < 1 \text{ and} \\ < 0 & \text{if } x > 1. \end{cases}$$

Hence, $f''(x) < 0$ for $x \neq 1$; $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$, finally $f(x) < 0$ for $x \neq 1$.

**Comment.** Version 2 is much more difficult, of rather A5 or A6 difficulty. The induction in Version 1 is rather straightforward, while all three above solutions of Version 2 require some creativity.

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**A2.** Let $\mathcal{A}$ denote the set of all polynomials in three variables $x, y, z$ with integer coefficients. Let $\mathcal{B}$ denote the subset of $\mathcal{A}$ formed by all polynomials which can be expressed as

$$(x+y+z)P(x,y,z) + (xy+yz+zx)Q(x,y,z) + xyzR(x,y,z)$$

with $P, Q, R \in \mathcal{A}$. Find the smallest non-negative integer $n$ such that $x^i y^j z^k \in \mathcal{B}$ for all non-negative integers $i, j, k$ satisfying $i + j + k \geqslant n$.

*(Venezuela)*

**Answer:** $n = 4$.

**Solution.** We start by showing that $n \leqslant 4$, i.e., any monomial $f = x^i y^j z^k$ with $i + j + k \geqslant 4$ belongs to $\mathcal{B}$. Assume that $i \geqslant j \geqslant k$, the other cases are analogous.

Let $x + y + z = p$, $xy + yz + zx = q$ and $xyz = r$. Then

$$0 = (x-x)(x-y)(x-z) = x^3 - px^2 + qx - r,$$

therefore $x^3 \in \mathcal{B}$. Next, $x^2y^2 = xyq - (x+y)r \in \mathcal{B}$.

If $k \geqslant 1$, then $r$ divides $f$, thus $f \in \mathcal{B}$. If $k = 0$ and $j \geqslant 2$, then $x^2y^2$ divides $f$, thus we have $f \in \mathcal{B}$ again. Finally, if $k = 0, j \leqslant 1$, then $x^3$ divides $f$ and $f \in \mathcal{B}$ in this case also.

In order to prove that $n \geqslant 4$, we show that the monomial $x^2y$ does not belong to $\mathcal{B}$. Assume the contrary:

$$x^2y = pP + qQ + rR \tag{1}$$

for some polynomials $P, Q, R$. If polynomial $P$ contains the monomial $x^2$ (with nonzero coefficient), then $pP + qQ + rR$ contains the monomial $x^3$ with the same nonzero coefficient. So $P$ does not contain $x^2$, $y^2$, $z^2$ and we may write

$$x^2y = (x+y+z)(axy + byz + czx) + (xy+yz+zx)(dx + ey + fz) + gxyz,$$

where $a, b, c$; $d, e, f$; $g$ are the coefficients of $xy, yz, zx$; $x, y, z$; $xyz$ in the polynomials $P$; $Q$; $R$, respectively (the remaining coefficients do not affect the monomials of degree 3 in $pP + qQ + rR$). By considering the coefficients of $xy^2$ we get $e = -a$, analogously $e = -b$, $f = -b$, $f = -c$, $d = -c$, thus $a = b = c$ and $f = e = d = -a$, but then the coefficient of $x^2y$ in the right hand side equals $a + d = 0 \neq 1$.

**Comment 1.** The general question is the following. Call a polynomial $f(x_1, \ldots, x_n)$ with integer coefficients *nice*, if $f(0, 0, \ldots, 0) = 0$ and $f(x_{\pi_1}, \ldots, x_{\pi_n}) = f(x_1, \ldots, x_n)$ for any permutation $\pi$ of $1, \ldots, n$ (in other words, $f$ is symmetric and its constant term is zero.) Denote by $\mathcal{I}$ the set of polynomials of the form

$$p_1q_1 + p_2q_2 + \ldots + p_mq_m, \tag{2}$$

where $m$ is an integer, $q_1, \ldots, q_m$ are polynomials with integer coefficients, and $p_1, \ldots, p_m$ are nice polynomials. Find the least $N$ for which any monomial of degree at least $N$ belongs to $\mathcal{I}$.

The **answer** is $n(n-1)/2 + 1$. The lower bound follows from the following claim: the polynomial

$$F(x_1, \ldots, x_n) = x_2x_3^2x_4^3 \cdot \ldots \cdot x_n^{n-1}$$

does not belong to $\mathcal{I}$.

Assume that $F = \sum p_iq_i$, according to (2). By taking only the monomials of degree $n(n-1)/2$, we can additionally assume that every $p_i$ and every $q_i$ is homogeneous, $\deg p_i > 0$, and $\deg p_i + \deg q_i = \deg F = n(n-1)/2$ for all $i$.

Consider the alternating sum

$$\sum_{\pi} \operatorname{sign}(\pi) F(x_{\pi_1}, \ldots, x_{\pi_n}) = \sum_{i=1}^{m} p_i \sum_{\pi} \operatorname{sign}(\pi) q_i(x_{\pi_1}, \ldots, x_{\pi_n}) := S, \tag{3}$$

where the summation is done over all permutations $\pi$ of $1, \dots n$, and $\operatorname{sign}(\pi)$ denotes the sign of the permutation $\pi$. Since $\deg q_i = n(n-1)/2 - \deg p_i < n(n-1)/2$, in any monomial $Q$ of $q_i$, there are at least two variables, say $x_\alpha$ and $x_\beta$, with equal exponents. Therefore $\sum_\pi \operatorname{sign}(\pi) Q(x_{\pi_1}, \dots, x_{\pi_n}) = 0$, because each pair of terms that corresponds to permutations which differ by the transposition of $\alpha$ and $\beta$, cancels out. This holds for any $i = 1, \dots, m$ and any monomial of $q_i$, so $S = 0$. But the left hand side of (3) is a non-zero polynomial. This is a contradiction.

Let us now prove, using induction on $n$, that any monomial $h = x_1^{c_1} \dots x_n^{c_n}$ of degree $n(n-1)/2+1$ belongs to $\mathcal{I}$, and additionally all $p_i, q_i$ in the representation (2) can be chosen homogeneous with sum of degrees equal to $n(n-1)/2+1$. (Obviously, any monomial of degree at least $n(n-1)/2+1$ is divisible by a monomial of degree exactly $n(n-1)/2+1$, thus this suffices.) The proposition is true for $n = 1$, so assume that $n > 1$ and that the proposition is proved for smaller values of $n$.

We proceed by an internal induction on $S := |\{i\colon c_i = 0\}|$. In the base case $S = 0$ the monomial $h$ is divisible by the nice polynomial $x_1 \cdot \ldots \cdot x_n$, therefore $h \in \mathcal{I}$. Now assume that $S > 0$ and that the claim holds for smaller values of $S$. Let $T = n - S$. We may assume that $c_{T+1} = \ldots = c_n = 0$ and $h = x_1 \cdot \ldots \cdot x_T g(x_1, \dots, x_{n-1})$, where $\deg g = n(n-1)/2 - T + 1 \geqslant (n-1)(n-2)/2 + 1$. Using the outer induction hypothesis we represent $g$ as $p_1 q_1 + \ldots + p_m q_m$, where $p_i(x_1, \dots, x_{n-1})$ are nice polynomials in $n-1$ variables. There exist nice homogeneous polynomials $P_i(x_1, \dots, x_n)$ such that $P_i(x_1, \dots, x_{n-1}, 0) = p_i(x_1, \dots, x_{n-1})$. In other words, $\Delta_i := p_i(x_1, \dots, x_{n-1}) - P_i(x_1, \dots, x_{n-1}, x_n)$ is divisible by $x_n$, let $\Delta_i = x_n g_i$. We get

$$h = x_1 \cdot \ldots \cdot x_T \sum p_i q_i = x_1 \cdot \ldots \cdot x_T \sum (P_i + x_n g_i) q_i = (x_1 \cdot \ldots \cdot x_T x_n) \sum g_i q_i + \sum P_i q_i \in \mathcal{I}.$$

The first term belongs to $\mathcal{I}$ by the inner induction hypothesis. This completes both inductions.

**Comment 2.** The solutions above work smoothly for the versions of the original problem and its extensions to the case of $n$ variables, where all polynomials are assumed to have real coefficients. In the version with integer coefficients, the argument showing that $x^2 y \notin \mathcal{B}$ can be simplified: it is not hard to show that in every polynomial $f \in \mathcal{B}$, the sum of the coefficients of $x^2y$, $x^2z$, $y^2x$, $y^2z$, $z^2x$ and $z^2y$ is even. A similar fact holds for any number of variables and also implies that $N \geqslant n(n-1)/2+1$ in terms of the previous comment.

**A3.** Suppose that $a$, $b$, $c$, $d$ are positive real numbers satisfying $(a+c)(b+d) = ac+bd$. Find the smallest possible value of

$$S = \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

*(Israel)*

**Answer:** The smallest possible value is 8.

**Solution 1.** To show that $S \geqslant 8$, apply the AM–GM inequality twice as follows:

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\left(\frac{b}{c}+\frac{d}{a}\right) \geqslant 2\sqrt{\frac{ac}{bd}}+2\sqrt{\frac{bd}{ac}} = \frac{2(ac+bd)}{\sqrt{abcd}} = \frac{2(a+c)(b+d)}{\sqrt{abcd}} \geqslant 2\cdot\frac{2\sqrt{ac}\cdot 2\sqrt{bd}}{\sqrt{abcd}} = 8\,.$$

The above inequalities turn into equalities when $a = c$ and $b = d$. Then the condition $(a+c)(b+d) = ac+bd$ can be rewritten as $4ab = a^2+b^2$. So it is satisfied when $a/b = 2 \pm \sqrt{3}$. Hence, $S$ attains value 8, e.g., when $a = c = 1$ and $b = d = 2+\sqrt{3}$.

**Solution 2.** By homogeneity we may suppose that $abcd = 1$. Let $ab = C$, $bc = A$ and $ca = B$. Then $a$, $b$, $c$ can be reconstructed from $A$, $B$ and $C$ as $a = \sqrt{BC/A}$, $b = \sqrt{AC/B}$ and $c = \sqrt{AB/C}$. Moreover, the condition $(a+c)(b+d) = ac+bd$ can be written in terms of $A, B, C$ as

$$A + \frac{1}{A} + C + \frac{1}{C} = bc + ad + ab + cd = (a+c)(b+d) = ac+bd = B + \frac{1}{B}.$$

We then need to minimize the expression

$$\begin{aligned} S :=& \frac{ad+bc}{bd} + \frac{ab+cd}{ac} = \left(A+\frac{1}{A}\right)B + \left(C+\frac{1}{C}\right)\frac{1}{B} \\ =& \left(A+\frac{1}{A}\right)\left(B-\frac{1}{B}\right) + \left(A+\frac{1}{A}+C+\frac{1}{C}\right)\frac{1}{B} \\ =& \left(A+\frac{1}{A}\right)\left(B-\frac{1}{B}\right) + \left(B+\frac{1}{B}\right)\frac{1}{B}. \end{aligned}$$

Without loss of generality assume that $B \geqslant 1$ (otherwise, we may replace $B$ by $1/B$ and swap $A$ and $C$, this changes neither the relation nor the function to be maximized). Therefore, we can write

$$S \geqslant 2\left(B-\frac{1}{B}\right) + \left(B+\frac{1}{B}\right)\frac{1}{B} = 2B + \left(1-\frac{1}{B}\right)^2 =: f(B).$$

Clearly, $f$ increases on $[1,\infty)$. Since

$$B + \frac{1}{B} = A + \frac{1}{A} + C + \frac{1}{C} \geqslant 4,$$

we have $B \geqslant B'$, where $B' = 2+\sqrt{3}$ is the unique root greater than 1 of the equation $B' + 1/B' = 4$. Hence,

$$S \geqslant f(B) \geqslant f(B') = 2\left(B'-\frac{1}{B'}\right) + \left(B'+\frac{1}{B'}\right)\frac{1}{B'} = 2B' - \frac{2}{B'} + \frac{4}{B'} = 8.$$

It remains to note that when $A = C = 1$ and $B = B'$ we have the equality $S = 8$.

**Solution 3.** We present another proof of the inequality $S \geqslant 8$. We start with the estimate

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\left(\frac{b}{c}+\frac{d}{a}\right) \geqslant 2\sqrt{\frac{ac}{bd}}+2\sqrt{\frac{bd}{ac}}.$$

Let $y=\sqrt{ac}$ and $z=\sqrt{bd}$, and assume, without loss of generality, that $ac \geqslant bd$. By the AM–GM inequality, we have

$$y^2+z^2=ac+bd=(a+c)(b+d) \geqslant 2\sqrt{ac}\cdot 2\sqrt{bd}=4yz.$$

Substituting $x=y/z$, we get $4x \leqslant x^2+1$. For $x \geqslant 1$, this holds if and only if $x \geqslant 2+\sqrt{3}$.

Now we have

$$2\sqrt{\frac{ac}{bd}}+2\sqrt{\frac{bd}{ac}}=2\left(x+\frac{1}{x}\right).$$

Clearly, this is minimized by setting $x(\geqslant 1)$ as close to 1 as possible, i.e., by taking $x=2+\sqrt{3}$. Then $2(x+1/x)=2((2+\sqrt{3})+(2-\sqrt{3}))=8$, as required.

**A4.** Let $a, b, c, d$ be four real numbers such that $a \geqslant b \geqslant c \geqslant d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)\, a^a\, b^b\, c^c\, d^d < 1.$$

*(Belgium)*

**Solution 1.** The weighted AM–GM inequality with weights $a, b, c, d$ gives

$$a^a b^b c^c d^d \leqslant a \cdot a + b \cdot b + c \cdot c + d \cdot d = a^2 + b^2 + c^2 + d^2,$$

so it suffices to prove that $(a + 2b + 3c + 4d)(a^2 + b^2 + c^2 + d^2) < 1 = (a + b + c + d)^3$. This can be done in various ways, for example:

$$\begin{aligned}(a + b + c + d)^3 &> a^2(a + 3b + 3c + 3d) + b^2(3a + b + 3c + 3d) \\ &\quad + c^2(3a + 3b + c + 3d) + d^2(3a + 3b + 3c + d) \\ &\geqslant (a^2 + b^2 + c^2 + d^2) \cdot (a + 2b + 3c + 4d).\end{aligned}$$

**Solution 2.** From $b \geqslant d$ we get

$$a + 2b + 3c + 4d \leqslant a + 3b + 3c + 3d = 3 - 2a.$$

If $a < \frac{1}{2}$, then the statement can be proved by

$$(a + 2b + 3c + 4d)\, a^a b^b c^c d^d \leqslant (3 - 2a) a^a a^b a^c a^d = (3 - 2a)a = 1 - (1 - a)(1 - 2a) < 1.$$

From now on we assume $\frac{1}{2} \leqslant a < 1$.

By $b, c, d < 1 - a$ we have

$$b^b c^c d^d < (1 - a)^b \cdot (1 - a)^c \cdot (1 - a)^d = (1 - a)^{1-a}.$$

Therefore,

$$(a + 2b + 3c + 4d) a^a b^b c^c d^d < (3 - 2a)\, a^a\, (1 - a)^{1-a}.$$

For $0 < x < 1$, consider the functions

$$f(x) = (3 - 2x) x^x (1 - x)^{1-x} \quad \text{and} \quad g(x) = \log f(x) = \log(3 - 2x) + x \log x + (1 - x) \log(1 - x);$$

hereafter, log denotes the natural logarithm. It is easy to verify that

$$g''(x) = -\frac{4}{(3 - 2x)^2} + \frac{1}{x} + \frac{1}{1 - x} = \frac{1 + 8(1 - x)^2}{x(1 - x)(3 - 2x)^2} > 0,$$

so $g$ is strictly convex on $(0, 1)$.

By $g\left(\frac{1}{2}\right) = \log 2 + 2 \cdot \frac{1}{2} \log \frac{1}{2} = 0$ and $\lim\limits_{x \to 1-} g(x) = 0$, we have $g(x) \leqslant 0$ (and hence $f(x) \leqslant 1$) for all $x \in \left[\frac{1}{2}, 1\right)$, and therefore

$$(a + 2b + 3c + 4d) a^a b^b c^c d^d < f(a) \leqslant 1.$$

**Comment.** For a large number of variables $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$ with $\sum_i a_i = 1$, the inequality

$$\left(\sum_i i a_i\right) \prod_i a_i^{a_i} \leqslant 1$$

does not necessarily hold. Indeed, let $a_2 = a_3 = \ldots = a_n = \varepsilon$ and $a_1 = 1 - (n - 1)\varepsilon$, where $n$ and $\varepsilon \in (0, 1/n)$ will be chosen later. Then

$$\left(\sum_i i a_i\right) \prod_i a_i^{a_i} = \left(1 + \frac{n(n - 1)}{2} \varepsilon\right) \varepsilon^{(n-1)\varepsilon} (1 - (n - 1)\varepsilon)^{1-(n-1)\varepsilon}. \tag{1}$$

If $\varepsilon = C/n^2$ with an arbitrary fixed $C > 0$ and $n \to \infty$, then the factors $\varepsilon^{(n-1)\varepsilon} = \exp((n - 1)\varepsilon \log \varepsilon)$ and $(1 - (n - 1)\varepsilon)^{1-(n-1)\varepsilon}$ tend to 1, so the limit of (1) in this set-up equals $1 + C/2$. This is not simply greater than 1, but it can be arbitrarily large.

**A5.** A magician intends to perform the following trick. She announces a positive integer $n$, along with $2n$ real numbers $x_1 < \ldots < x_{2n}$, to the audience. A member of the audience then secretly chooses a polynomial $P(x)$ of degree $n$ with real coefficients, computes the $2n$ values $P(x_1), \ldots, P(x_{2n})$, and writes down these $2n$ values on the blackboard in non-decreasing order. After that the magician announces the secret polynomial to the audience.

Can the magician find a strategy to perform such a trick?

*(Luxembourg)*

**Answer:** No, she cannot.

**Solution.** Let $x_1 < x_2 < \ldots < x_{2n}$ be real numbers chosen by the magician. We will construct two distinct polynomials $P(x)$ and $Q(x)$, each of degree $n$, such that the member of audience will write down the same sequence for both polynomials. This will mean that the magician cannot distinguish $P$ from $Q$.

*Claim.* There exists a polynomial $P(x)$ of degree $n$ such that $P(x_{2i-1}) + P(x_{2i}) = 0$ for $i = 1, 2, \ldots, n$.

*Proof.* We want to find a polynomial $a_n x^n + \ldots + a_1 x + a_0$ satisfying the following system of equations:

$$\begin{cases} (x_1^n + x_2^n)a_n + (x_1^{n-1} + x_2^{n-1})a_{n-1} + \ldots + 2a_0 = 0 \\ (x_3^n + x_4^n)a_n + (x_3^{n-1} + x_4^{n-1})a_{n-1} + \ldots + 2a_0 = 0 \\ \ldots \\ (x_{2n-1}^n + x_{2n}^n)a_n + (x_{2n-1}^{n-1} + x_{2n}^{n-1})a_{n-1} + \ldots + 2a_0 = 0 \end{cases}$$

We use the well known fact that a homogeneous system of $n$ linear equations in $n+1$ variables has a nonzero solution. (This fact can be proved using induction on $n$, via elimination of variables.) Applying this fact to the above system, we find a nonzero polynomial $P(x)$ of degree not exceeding $n$ such that its coefficients $a_0, \ldots, a_n$ satisfy this system. Therefore $P(x_{2i-1}) + P(x_{2i}) = 0$ for all $i = 1, 2, \ldots, n$. Notice that $P$ has a root on each segment $[x_{2i-1}, x_{2i}]$ by the Intermediate Value theorem, so $n$ roots in total. Since $P$ is nonzero, we get $\deg P = n$. □

Now consider a polynomial $P(x)$ provided by the Claim, and take $Q(x) = -P(x)$. The properties of $P(x)$ yield that $P(x_{2i-1}) = Q(x_{2i})$ and $Q(x_{2i-1}) = P(x_{2i})$ for all $i = 1, 2, \ldots, n$. It is also clear that $P \neq -P = Q$ and $\deg Q = \deg P = n$.

**Comment.** It can be shown that for any positive integer $n$ the magician can choose $2n+1$ distinct real numbers so as to perform such a trick. Moreover, she can perform such a trick with almost all (in a proper sense) $(2n+1)$-tuples of numbers.

**A6.** Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$f^{a^2+b^2}(a+b) = af(a) + bf(b) \qquad \text{for every } a, b \in \mathbb{Z}.$$

Here, $f^n$ denotes the $n^{\text{th}}$ iteration of $f$, i.e., $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for all $n \geqslant 0$.

*(Slovakia)*

**Answer:** Either $f(x) = 0$ for all $x \in \mathbb{Z}$, or $f(x) = x + 1$ for all $x \in \mathbb{Z}$.

**Solution.** Refer to the main equation as $E(a, b)$.

$E(0, b)$ reads as $f^{b^2}(b) = bf(b)$. For $b = -1$ this gives $f(-1) = 0$.

Now $E(a, -1)$ reads as

$$f^{a^2+1}(a-1) = af(a) = f^{a^2}(a). \tag{1}$$

For $x \in \mathbb{Z}$ define the *orbit of* $x$ by $\mathcal{O}(x) = \{x, f(x), f(f(x)), \ldots\} \subseteq \mathbb{Z}$. We see that the orbits $\mathcal{O}(a-1)$ and $\mathcal{O}(a)$ differ by finitely many terms. Hence, any two orbits differ by finitely many terms. In particular, this implies that either all orbits are finite or all orbits are infinite.

*Case 1: All orbits are finite.*

Then $\mathcal{O}(0)$ is finite. Using $E(a, -a)$ we get

$$a\big(f(a) - f(-a)\big) = af(a) - af(-a) = f^{2a^2}(0) \in \mathcal{O}(0).$$

For $|a| > \max\limits_{z \in \mathcal{O}(0)} |z|$, this yields $f(a) = f(-a)$ and $f^{2a^2}(0) = 0$. Therefore, the sequence $\big(f^k(0)\colon k = 0, 1, \ldots\big)$ is purely periodic with a minimal period $T$ which divides $2a^2$. Analogously, $T$ divides $2(a+1)^2$, therefore, $T \mid \gcd\big(2a^2, 2(a+1)^2\big) = 2$, i.e., $f(f(0)) = 0$ and $a\big(f(a) - f(-a)\big) = f^{2a^2}(0) = 0$ for all $a$. Thus,

$$f(a) = f(-a) \qquad \text{for all } a \neq 0; \tag{♣}$$

$$\text{in particular,} \quad f(1) = f(-1) = 0. \tag{♠}$$

Next, for each $n \in \mathbb{Z}$, by $E(n, 1-n)$ we get

$$nf(n) + (1-n)f(1-n) = f^{n^2+(1-n)^2}(1) = f^{2n^2-2n}(0) = 0. \tag{♡}$$

Assume that there exists some $m \neq 0$ such that $f(m) \neq 0$. Choose such an $m$ for which $|m|$ is minimal possible. Then $|m| > 1$ due to (♠); $f(|m|) \neq 0$ due to (♣); and $f(1 - |m|) \neq 0$ due to (♡) for $n = |m|$. This contradicts to the minimality assumption.

So, $f(n) = 0$ for $n \neq 0$. Finally, $f(0) = f^3(0) = f^4(2) = 2f(2) = 0$. Clearly, the function $f(x) \equiv 0$ satisfies the problem condition, which provides the first of the two answers.

*Case 2: All orbits are infinite.*

Since the orbits $\mathcal{O}(a)$ and $\mathcal{O}(a-1)$ differ by finitely many terms for all $a \in \mathbb{Z}$, each two orbits $\mathcal{O}(a)$ and $\mathcal{O}(b)$ have infinitely many common terms for arbitrary $a, b \in \mathbb{Z}$.

For a minute, fix any $a, b \in \mathbb{Z}$. We claim that all pairs $(n, m)$ of nonnegative integers such that $f^n(a) = f^m(b)$ have the same difference $n - m$. Arguing indirectly, we have $f^n(a) = f^m(b)$ and $f^p(a) = f^q(b)$ with, say, $n - m > p - q$, then $f^{p+m+k}(b) = f^{p+n+k}(a) = f^{q+n+k}(b)$, for all nonnegative integers $k$. This means that $f^{\ell+(n-m)-(p-q)}(b) = f^\ell(b)$ for all sufficiently large $\ell$, i.e., that the sequence $\big(f^n(b)\big)$ is eventually periodic, so $\mathcal{O}(b)$ is finite, which is impossible.

Now, for every $a, b \in \mathbb{Z}$, denote the common difference $n - m$ defined above by $X(a, b)$. We have $X(a-1, a) = 1$ by (1). Trivially, $X(a, b) + X(b, c) = X(a, c)$, as if $f^n(a) = f^m(b)$ and $f^p(b) = f^q(c)$, then $f^{p+n}(a) = f^{p+m}(b) = f^{q+m}(c)$. These two properties imply that $X(a, b) = b - a$ for all $a, b \in \mathbb{Z}$.

But (1) yields $f^{a^2+1}(f(a-1)) = f^{a^2}(f(a))$, so

$$1 = X\big(f(a-1), f(a)\big) = f(a) - f(a-1) \quad \text{for all } a \in \mathbb{Z}.$$

Recalling that $f(-1) = 0$, we conclude by (two-sided) induction on $x$ that $f(x) = x + 1$ for all $x \in \mathbb{Z}$.

Finally, the obtained function also satisfies the assumption. Indeed, $f^n(x) = x + n$ for all $n \geqslant 0$, so

$$f^{a^2+b^2}(a+b) = a + b + a^2 + b^2 = af(a) + bf(b).$$

**Comment.** There are many possible variations of the solution above, but it seems that finiteness of orbits seems to be a crucial distinction in all solutions. However, the case distinction could be made in different ways; in particular, there exist some versions of Case 1 which work whenever there is at least *one* finite orbit.

We believe that Case 2 is conceptually harder than Case 1.

**A7.** Let $n$ and $k$ be positive integers. Prove that for $a_1, \ldots, a_n \in [1, 2^k]$ one has

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{a_1^2 + \ldots + a_i^2}} \leqslant 4\sqrt{kn}.$$

*(Iran)*

**Solution 1.** Partition the set of indices $\{1, 2, \ldots, n\}$ into disjoint subsets $M_1, M_2, \ldots, M_k$ so that $a_\ell \in [2^{j-1}, 2^j]$ for $\ell \in M_j$. Then, if $|M_j| =: p_j$, we have

$$\sum_{\ell \in M_j} \frac{a_\ell}{\sqrt{a_1^2 + \ldots + a_\ell^2}} \leqslant \sum_{i=1}^{p_j} \frac{2^j}{2^{j-1}\sqrt{i}} = 2\sum_{i=1}^{p_j} \frac{1}{\sqrt{i}},$$

where we used that $a_\ell \leqslant 2^j$ and in the denominator every index from $M_j$ contributes at least $(2^{j-1})^2$. Now, using $\sqrt{i} - \sqrt{i-1} = \frac{1}{\sqrt{i}+\sqrt{i-1}} \geqslant \frac{1}{2\sqrt{i}}$, we deduce that

$$\sum_{\ell \in M_j} \frac{a_\ell}{\sqrt{a_1^2 + \ldots + a_\ell^2}} \leqslant 2\sum_{i=1}^{p_j} \frac{1}{\sqrt{i}} \leqslant 2\sum_{i=1}^{p_j} 2(\sqrt{i} - \sqrt{i-1}) = 4\sqrt{p_j}.$$

Therefore, summing over $j = 1, \ldots, k$ and using the QM–AM inequality, we obtain

$$\sum_{\ell=1}^{n} \frac{a_\ell}{\sqrt{a_1^2 + \ldots + a_\ell^2}} \leqslant 4\sum_{j=1}^{k} \sqrt{|M_j|} \leqslant 4\sqrt{k\sum_{j=1}^{k} |M_j|} = 4\sqrt{kn}.$$

**Comment.** Consider the function $f(a_1, \ldots, a_n) = \sum_{i=1}^{n} \frac{a_i}{\sqrt{a_1^2+\ldots+a_i^2}}$. One can see that rearranging the variables in increasing order can only increase the value of $f(a_1, \ldots, a_n)$. Indeed, if $a_j > a_{j+1}$ for some index $j$ then we have

$$f(a_1, \ldots, a_{j-1}, a_{j+1}, a_j, a_{j+2}, \ldots, a_n) - f(a_1, \ldots, a_n) = \frac{a}{S} + \frac{b}{\sqrt{S^2 - a^2}} - \frac{b}{S} - \frac{a}{\sqrt{S^2 - b^2}}$$

where $a = a_j, b = a_{j+1}$, and $S = \sqrt{a_1^2 + \ldots + a_{j+1}^2}$. The positivity of the last expression above follows from

$$\frac{b}{\sqrt{S^2 - a^2}} - \frac{b}{S} = \frac{a^2 b}{S\sqrt{S^2 - a^2} \cdot (S + \sqrt{S^2 - a^2})} > \frac{ab^2}{S\sqrt{S^2 - b^2} \cdot (S + \sqrt{S^2 - b^2})} = \frac{a}{\sqrt{S^2 - b^2}} - \frac{a}{S}.$$

**Comment.** If $k < n$, the example $a_m := 2^{k(m-1)/n}$ shows that the problem statement is sharp up to a multiplicative constant. For $k \geqslant n$ the trivial upper bound $n$ becomes sharp up to a multiplicative constant.

**Solution 2.** Apply induction on $n$. The base $n \leqslant 16$ is clear: our sum does not exceed $n \leqslant 4\sqrt{nk}$. For the inductive step from $1, \ldots, n-1$ to $n \geqslant 17$ consider two similar cases.

*Case 1: $n = 2t$.*

Let $x_\ell = \frac{a_\ell}{\sqrt{a_1^2+\ldots+a_\ell^2}}$. We have

$$\exp(-x_{t+1}^2 - \ldots - x_{2t}^2) \geqslant \left(1 - x_{t+1}^2\right) \ldots \left(1 - x_{2t}^2\right) = \frac{a_1^2 + \ldots + a_t^2}{a_1^2 + \ldots + a_{2t}^2} \geqslant \frac{1}{1 + 4^k},$$

where we used that the product is telescopic and then an estimate $a_{t+i} \leqslant 2^k a_i$ for $i = 1, \ldots, t$. Therefore, $x_{t+1}^2 + \ldots + x_{2t}^2 \leqslant \log(4^k + 1) \leqslant 2k$, where log denotes the natural logarithm. This implies $x_{t+1} + \ldots + x_{2t} \leqslant \sqrt{2kt}$. Hence, using the inductive hypothesis for $n = t$ we get

$$\sum_{\ell=1}^{2t} x_\ell \leqslant 4\sqrt{kt} + \sqrt{2kt} \leqslant 4\sqrt{2kt}.$$

*Case 2:* $n = 2t + 1$.

Analogously, we get $x_{t+2}^2 + \ldots + x_{2t+1}^2 \leqslant \log(4^k + 1) \leqslant 2k$ and

$$\sum_{\ell=1}^{2t+1} x_\ell \leqslant 4\sqrt{k(t+1)} + \sqrt{2kt} \leqslant 4\sqrt{k(2t+1)}.$$

The last inequality is true for all $t \geqslant 8$ since

$$4\sqrt{2t+1} - \sqrt{2t} \geqslant 3\sqrt{2t} = \sqrt{18t} \geqslant \sqrt{16t+16} = 4\sqrt{t+1}.$$

**A8.** Let $\mathbb{R}^+$ be the set of positive real numbers. Determine all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all positive real numbers $x$ and $y$,

$$f\big(x + f(xy)\big) + y = f(x)f(y) + 1. \tag{$*$}$$

*(Ukraine)*

**Answer:** $f(x) = x + 1$.

**Solution 1.** A straightforward check shows that $f(x) = x + 1$ satisfies $(*)$. We divide the proof of the converse statement into a sequence of steps.

*Step 1: $f$ is injective.*

Put $x = 1$ in $(*)$ and rearrange the terms to get

$$y = f(1)f(y) + 1 - f\big(1 + f(y)\big).$$

Therefore, if $f(y_1) = f(y_2)$, then $y_1 = y_2$.

*Step 2: $f$ is (strictly) monotone increasing.*

For any fixed $y \in \mathbb{R}^+$, the function

$$g(x) := f\big(x + f(xy)\big) = f(x)f(y) + 1 - y$$

is injective by Step 1. Therefore, $x_1 + f(x_1y) \neq x_2 + f(x_2y)$ for all $y, x_1, x_2 \in \mathbb{R}^+$ with $x_1 \neq x_2$. Plugging in $z_i = x_iy$, we arrive at

$$\frac{z_1 - z_2}{y} \neq f(z_2) - f(z_1), \quad \text{or} \quad \frac{1}{y} \neq \frac{f(z_2) - f(z_1)}{z_1 - z_2}$$

for all $y, z_1, z_2 \in \mathbb{R}^+$ with $z_1 \neq z_2$. This means that the right-hand side of the rightmost relation is always non-positive, i.e., $f$ is monotone non-decreasing. Since $f$ is injective, it is strictly monotone.

*Step 3: There exist constants $a$ and $b$ such that $f(y) = ay + b$ for all $y \in \mathbb{R}^+$.*

Since $f$ is monotone and bounded from below by 0, for each $x_0 \geqslant 0$, there exists a right limit $\lim_{x \searrow x_0} f(x) \geqslant 0$. Put $p = \lim_{x \searrow 0} f(x)$ and $q = \lim_{x \searrow p} f(x)$.

Fix an arbitrary $y$ and take the limit of $(*)$ as $x \searrow 0$. We have $f(xy) \searrow p$ and hence $f\big(x + f(xy)\big) \searrow q$; therefore, we obtain

$$q + y = pf(y) + 1, \quad \text{or} \quad f(y) = \frac{q + y - 1}{p}.$$

(Notice that $p \neq 0$, otherwise $q + y = 1$ for all $y$, which is absurd.) The claim is proved.

*Step 4: $f(x) = x + 1$ for all $x \in \mathbb{R}^+$.*

Based on the previous step, write $f(x) = ax + b$. Putting this relation into $(*)$ we get

$$a\,(x + axy + b) + b + y = (ax + b)(ay + b) + 1,$$

which can be rewritten as

$$(a - ab)x + (1 - ab)y + ab + b - b^2 - 1 = 0 \qquad \text{for all } x, y \in \mathbb{R}^+.$$

This identity may hold only if all the coefficients are 0, i.e.,

$$a - ab = 1 - ab = ab + b - b^2 - 1 = 0.$$

Hence, $a = b = 1$.

**Solution 2.** We provide another proof that $f(x) = x + 1$ is the only function satisfying $(*)$.

Put $a = f(1)$. Define the function $\phi\colon \mathbb{R}^+ \to \mathbb{R}$ by

$$\phi(x) = f(x) - x - 1.$$

Then equation $(*)$ reads as

$$\phi(x + f(xy)) = f(x)f(y) - f(xy) - x - y. \tag{1}$$

Since the right-hand side of (1) is symmetric under swapping $x$ and $y$, we obtain

$$\phi\big(x + f(xy)\big) = \phi\big(y + f(xy)\big).$$

In particular, substituting $(x, y) = (t, 1/t)$ we get

$$\phi(a + t) = \phi\left(a + \frac{1}{t}\right), \qquad t \in \mathbb{R}^+. \tag{2}$$

Notice that the function $f$ is bounded from below by a positive constant. Indeed, for each $y \in \mathbb{R}^+$, the relation $(*)$ yields $f(x)f(y) > y - 1$, hence

$$f(x) > \frac{y-1}{f(y)} \qquad \text{for all } x \in \mathbb{R}^+.$$

If $y > 1$, this provides a desired positive lower bound for $f(x)$.

Now, let $M = \inf\limits_{x \in \mathbb{R}^+} f(x) > 0$. Then, for all $y \in \mathbb{R}^+$,

$$M \geqslant \frac{y-1}{f(y)}, \quad \text{or} \quad f(y) \geqslant \frac{y-1}{M}. \tag{3}$$

*Lemma 1.* The function $f(x)$ (and hence $\phi(x)$) is bounded on any segment $[p, q]$, where $0 < p < q < +\infty$.

*Proof.* $f$ is bounded from below by $M$. It remains to show that $f$ is bounded from above on $[p, q]$. Substituting $y = 1$ into $(*)$, we get

$$f\big(x + f(x)\big) = af(x). \tag{4}$$

Take $z \in [p, q]$ and put $s = f(z)$. By (4), we have

$$f(z + s) = as \quad \text{and} \quad f(z + s + as) = f\big(z + s + f(z + s)\big) = a^2 s.$$

Plugging in $(x, y) = \big(z, 1 + \frac{s}{z}\big)$ to $(*)$ and using (3), we obtain

$$f(z + as) = f\big(z + f(z + s)\big) = sf\left(1 + \frac{s}{z}\right) - \frac{s}{z} \geqslant \frac{s^2}{Mz} - \frac{s}{z}.$$

Now, substituting $(x, y) = \big(z + as, \frac{z}{z+as}\big)$ to $(*)$ and applying the above estimate and the estimate $f(y) \geqslant M$, we obtain

$$\begin{aligned} a^2 s = f(z + s + as) = f\big(z + as + f(z)\big) = f(z + as)f\left(\frac{z}{z + as}\right) + 1 - \frac{z}{z + as} \\ \geqslant Mf(z + as) \geqslant \frac{s^2}{z} - \frac{Ms}{z} \geqslant \frac{s^2}{q} - \frac{Ms}{p}. \end{aligned}$$

This yields $s \leqslant q\big(\frac{M}{p} + a^2\big) =: L$, and $f$ is bounded from above by $L$ on $[p, q]$. □

Applying Lemma 1 to the segment $[a, a+1]$, we see that $\phi$ is bounded on it. By (2) we get that $\phi$ is also bounded on $[a+1, +\infty)$, and hence on $[a, +\infty)$. Put $C = \max\{a, 3\}$.

*Lemma 2.* For all $x \geqslant C$, we have $\phi(x) = 0$ (and hence $f(x) = x + 1$).

*Proof.* Substituting $y = x$ to (1), we obtain

$$\phi\big(x + f(x^2)\big) = f(x)^2 - f(x^2) - 2x,$$

hence,

$$\phi\big(x + f(x^2)\big) + \phi(x^2) = f(x)^2 - (x+1)^2 = \phi(x)\big(f(x) + x + 1\big). \tag{5}$$

Since $f(x) + x + 1 \geqslant C + 1 \geqslant 4$, we obtain that

$$|\phi(x)| \leqslant \frac{1}{4}\left(\big|\phi\big(x + f(x^2)\big)\big| + \big|\phi(x^2)\big|\right). \tag{6}$$

Since $C \geqslant a$, there exists a finite supremum $S = \sup\limits_{x \geqslant C} |\phi(x)|$. For each $x \in [C, +\infty)$, both $x + f(x^2)$ and $x^2$ are greater than $x$; hence they also lie in $[C, +\infty)$. Therefore, taking the supremum of the left-hand side of (6) over $x \in [C, +\infty)$, we obtain $S \leqslant S/2$ and hence $S = 0$. Thus, $\phi(x) = 0$ for all $x \geqslant C$. □

It remains to show that $f(y) = y + 1$ when $0 < y < C$. For each $y$, choose $x > \max\big\{C, \frac{C}{y}\big\}$. Then all three numbers $x$, $xy$, and $x + f(xy)$ are greater than $C$, so $(*)$ reads as

$$(x + xy + 1) + 1 + y = (x+1)f(y) + 1, \quad \text{hence} \quad f(y) = y + 1.$$

**Comment 1.** It may be useful to rewrite $(*)$ in the form

$$\phi\big(x + f(xy)\big) + \phi(xy) = \phi(x)\phi(y) + x\phi(y) + y\phi(x) + \phi(x) + \phi(y).$$

This general identity easily implies both (1) and (5).

**Comment 2.** There are other ways to prove that $f(x) \geqslant x + 1$. Once one has proved this, they can use this stronger estimate instead of (3) in the proof of Lemma 1. Nevertheless, this does not make this proof simpler. So proving that $f(x) \geqslant x + 1$ does not seem to be a serious progress towards the solution of the problem. In what follows, we outline one possible proof of this inequality.

First of all, we improve inequality (3) by noticing that, in fact, $f(x)f(y) \geqslant y - 1 + M$, and hence

$$f(y) \geqslant \frac{y-1}{M} + 1. \tag{7}$$

Now we divide the argument into two steps.

*Step 1: We show that $M \leqslant 1$.*

Suppose that $M > 1$; recall the notation $a = f(1)$. Substituting $y = 1/x$ in $(*)$, we get

$$f(x + a) = f(x)f\left(\frac{1}{x}\right) + 1 - \frac{1}{x} \geqslant Mf(x),$$

provided that $x \geqslant 1$. By a straightforward induction on $\lceil (x-1)/a \rceil$, this yields

$$f(x) \geqslant M^{(x-1)/a}. \tag{8}$$

Now choose an arbitrary $x_0 \in \mathbb{R}^+$ and define a sequence $x_0, x_1, \ldots$ by $x_{n+1} = x_n + f(x_n) \geqslant x_n + M$ for all $n \geqslant 0$; notice that the sequence is unbounded. On the other hand, by (4) we get

$$ax_{n+1} > af(x_n) = f(x_{n+1}) \geqslant M^{(x_{n+1}-1)/a},$$

which cannot hold when $x_{n+1}$ is large enough.

*Step 2: We prove that $f(y) \geqslant y+1$ for all $y \in \mathbb{R}^+$.*

Arguing indirectly, choose $y \in \mathbb{R}^+$ such that $f(y) < y+1$, and choose $\mu$ with $f(y) < \mu < y+1$. Define a sequence $x_0, x_1, \ldots$ by choosing a large $x_0 \geqslant 1$ and setting $x_{n+1} = x_n + f(x_n y) \geqslant x_n + M$ for all $n \geqslant 0$ (this sequence is also unbounded). If $x_0$ is large enough, then (7) implies that $\big(\mu - f(y)\big) f(x_n) \geqslant 1 - y$ for all $n$. Therefore,

$$f(x_{n+1}) = f(y) f(x_n) + 1 - y \leqslant \mu f(x_n).$$

On the other hand, since $M \leqslant 1$, inequality (7) implies that $f(z) \geqslant z$, provided that $z \geqslant 1$. Hence, if $x_0$ is large enough, we have $x_{n+1} \geqslant x_n(1+y)$ for all $n$. Therefore,

$$x_0(1+y)^n \leqslant x_n \leqslant f(x_n) \leqslant \mu^n f(x_0),$$

which cannot hold when $n$ is large enough.

# Combinatorics

**C1.** Let $n$ be a positive integer. Find the number of permutations $a_1, a_2, \ldots, a_n$ of the sequence $1, 2, \ldots, n$ satisfying

$$a_1 \leqslant 2a_2 \leqslant 3a_3 \leqslant \ldots \leqslant na_n. \qquad (*)$$

*(United Kingdom)*

**Answer:** The number of such permutations is $F_{n+1}$, where $F_k$ is the $k^{\text{th}}$ Fibonacci number: $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$.

**Solution 1.** Denote by $P_n$ the number of permutations that satisfy $(*)$. It is easy to see that $P_1 = 1$ and $P_2 = 2$.

*Lemma 1.* Let $n \geqslant 3$. If a permutation $a_1, \ldots, a_n$ satisfies $(*)$ then either $a_n = n$, or $a_{n-1} = n$ and $a_n = n - 1$.

*Proof.* Let $k$ be the index for which $a_k = n$. If $k = n$ then we are done.

If $k = n - 1$ then, by $(*)$, we have $n(n-1) = (n-1)a_{n-1} \leqslant na_n$, so $a_n \geqslant n - 1$. Since $a_n \neq a_{n-1} = n$, the only choice for $a_n$ is $a_n = n - 1$.

Now suppose that $k \leqslant n - 2$. For every $k < i < n$ we have $kn = ka_k \leqslant ia_i < na_i$, so $a_i \geqslant k + 1$. Moreover, $na_n \geqslant (n-1)a_{n-1} \geqslant (n-1)(k+1) = nk + (n-1-k) > nk$, so $a_n \geqslant k + 1$. Now the $n - k + 1$ numbers $a_k, a_{k+1}, \ldots, a_n$ are all greater than $k$; but there are only $n - k$ such values; this is not possible. □

If $a_n = n$ then $a_1, a_2, \ldots, a_{n-1}$ must be a permutation of the numbers $1, \ldots, n-1$ satisfying $a_1 \leqslant 2a_2 \leqslant \ldots \leqslant (n-1)a_{n-1}$; there are $P_{n-1}$ such permutations. The last inequality in $(*)$, $(n-1)a_{n-1} \leqslant na_n = n^2$, holds true automatically.

If $(a_{n-1}, a_n) = (n, n-1)$, then $a_1, \ldots, a_{n-2}$ must be a permutation of $1, \ldots, n-2$ satisfying $a_1 \leqslant \ldots \leqslant (n-2)a_{n-2}$; there are $P_{n-2}$ such permutations. The last two inequalities in $(*)$ hold true automatically by $(n-2)a_{n-2} \leqslant (n-2)^2 < n(n-1) = (n-1)a_{n-1} = na_n$.

Hence, the sequence $(P_1, P_2, \ldots)$ satisfies the recurrence relation $P_n = P_{n-1} + P_{n-2}$ for $n \geqslant 3$. The first two elements are $P_1 = F_2$ and $P_2 = F_3$, so by a trivial induction we have $P_n = F_{n+1}$.

**Solution 2.** We claim that all sought permutations are of the following kind. Split $\{1, 2, \ldots, n\}$ into singletons and pairs of adjacent numbers. In each pair, swap the two numbers and keep the singletons unchanged.

Such permutations correspond to tilings of a $1 \times n$ chessboard using dominoes and unit squares; it is well-known that the number of such tilings is the Fibonacci number $F_{n+1}$.

The claim follows by induction from

*Lemma 2.* Assume that $a_1, \ldots, a_n$ is a permutation satisfying $(*)$, and $k$ is an integer such that $1 \leqslant k \leqslant n$ and $\{a_1, a_2, \ldots, a_{k-1}\} = \{1, 2, \ldots, k-1\}$. (If $k = 1$, the condition is empty.) Then either $a_k = k$, or $a_k = k + 1$ and $a_{k+1} = k$.

*Proof.* Choose $t$ with $a_t = k$. Since $k \notin \{a_1, \ldots, a_{k-1}\}$, we have either $t = k$ or $t > k$. If $t = k$ then we are done, so assume $t > k$.

Notice that one of the numbers among the $t - k$ numbers $a_k, a_{k+1}, \ldots, a_{t-1}$ is at least $t$, because there are only $t - k - 1$ values between $k$ and $t$. Let $i$ be an index with $k \leqslant i < t$ and $a_i \geqslant t$; then $kt = ta_t \geqslant ia_i \geqslant it \geqslant kt$, so that all the inequalities turn into equalities, hence $i = k$ and $a_k = t$. If $t = k + 1$, we are done.

Suppose that $t > k + 1$. Then the chain of inequalities $kt = ka_k \leqslant \ldots \leqslant ta_t = kt$ should also turn into a chain of equalities. From this point we can find contradictions in several ways; for example by pointing to $a_{t-1} = \frac{kt}{t-1} = k + \frac{k}{t-1}$ which cannot be an integer, or considering

the product of the numbers $(k+1)a_{k+1}, \ldots, (t-1)a_{t-1}$; the numbers $a_{k+1}, \ldots, a_{t-1}$ are distinct and greater than $k$, so

$$(kt)^{t-k-1} = (k+1)a_{k+1} \cdot (k+2)a_{k+2} \cdot \ldots \cdot (t-1)a_{t-1} \geqslant \big((k+1)(k+2) \cdot \ldots \cdot (t-1)\big)^2.$$

Notice that $(k+i)(t-i) = kt + i(t-k-i) > kt$ for $1 \leqslant i < t-k$. This leads to the contradiction

$$(kt)^{t-k-1} \geqslant \big((k+1)(k+2) \cdot \ldots \cdot (t-1)\big)^2 = \prod_{i=1}^{t-k-1} (k+i)(t-i) > (kt)^{t-k-1}.$$

Therefore, the case $t > k+1$ is not possible. □

**C2.** In a regular 100-gon, 41 vertices are colored black and the remaining 59 vertices are colored white. Prove that there exist 24 convex quadrilaterals $Q_1, \ldots, Q_{24}$ whose corners are vertices of the 100-gon, so that

- the quadrilaterals $Q_1, \ldots, Q_{24}$ are pairwise disjoint, and
- every quadrilateral $Q_i$ has three corners of one color and one corner of the other color.

*(Austria)*

**Solution.** Call a quadrilateral *skew-colored*, if it has three corners of one color and one corner of the other color. We will prove the following

*Claim.* If the vertices of a convex $(4k+1)$-gon $P$ are colored black and white such that each color is used at least $k$ times, then there exist $k$ pairwise disjoint skew-colored quadrilaterals whose vertices are vertices of $P$. (One vertex of $P$ remains unused.)

The problem statement follows by removing 3 arbitrary vertices of the 100-gon and applying the Claim to the remaining 97 vertices with $k = 24$.

*Proof of the Claim.* We prove by induction. For $k = 1$ we have a pentagon with at least one black and at least one white vertex. If the number of black vertices is even then remove a black vertex; otherwise remove a white vertex. In the remaining quadrilateral, there are an odd number of black and an odd number of white vertices, so the quadrilateral is skew-colored.

For the induction step, assume $k \geqslant 2$. Let $b$ and $w$ be the numbers of black and white vertices, respectively; then $b, w \geqslant k$ and $b + w = 4k + 1$. Without loss of generality we may assume $w \geqslant b$, so $k \leqslant b \leqslant 2k$ and $2k + 1 \leqslant w \leqslant 3k + 1$.

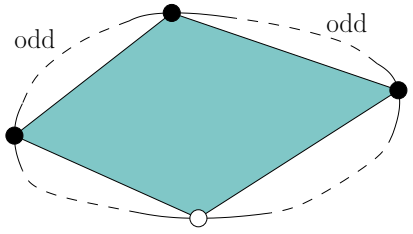
We want to find four consecutive vertices such that three of them are white, the fourth one is black. Denote the vertices by $V_1, V_2, \ldots, V_{4k+1}$ in counterclockwise order, such that $V_{4k+1}$ is black, and consider the following $k$ groups of vertices:

$$(V_1, V_2, V_3, V_4), (V_5, V_6, V_7, V_8), \ldots, (V_{4k-3}, V_{4k-2}, V_{4k-1}, V_{4k})$$

In these groups there are $w$ white and $b - 1$ black vertices. Since $w > b - 1$, there is a group, $(V_i, V_{i+1}, V_{i+2}, V_{i+3})$ that contains more white than black vertices. If three are white and one is black in that group, we are done. Otherwise, if $V_i, V_{i+1}, V_{i+2}, V_{i+3}$ are all white then let $V_j$ be the first black vertex among $V_{i+4}, \ldots, V_{4k+1}$ (recall that $V_{4k+1}$ is black); then $V_{j-3}$, $V_{j-2}$ and $V_{j-1}$ are white and $V_j$ is black.

Now we have four consecutive vertices $V_i, V_{i+1}, V_{i+2}, V_{i+3}$ that form a skew-colored quadrilateral. The remaining vertices form a convex $(4k-3)$-gon; $w - 3$ of them are white and $b - 1$ are black. Since $b - 1 \geqslant k - 1$ and $w - 3 \geqslant (2k + 1) - 3 > k - 1$, we can apply the Claim with $k - 1$. □

**Comment.** It is not true that the vertices of the 100-gon can be split into 25 skew-colored quadrilaterals. A possible counter-example is when the vertices $V_1, V_3, V_5, \ldots, V_{81}$ are black and the other vertices, $V_2, V_4, \ldots, V_{80}$ and $V_{82}, V_{83}, \ldots, V_{100}$ are white. For having 25 skew-colored quadrilaterals, there should be 8 containing three black vertices. But such a quadrilateral splits the other 96 vertices into four sets in such a way that at least two sets contain odd numbers of vertices and therefore they cannot be grouped into disjoint quadrilaterals.



**C3.** Let $n$ be an integer with $n \geqslant 2$. On a slope of a mountain, $n^2$ checkpoints are marked, numbered from 1 to $n^2$ from the bottom to the top. Each of two cable car companies, $A$ and $B$, operates $k$ cable cars numbered from 1 to $k$; each cable car provides a transfer from some checkpoint to a higher one. For each company, and for any $i$ and $j$ with $1 \leqslant i < j \leqslant k$, the starting point of car $j$ is higher than the starting point of car $i$; similarly, the finishing point of car $j$ is higher than the finishing point of car $i$. Say that two checkpoints are *linked* by some company if one can start from the lower checkpoint and reach the higher one by using one or more cars of that company (no movement on foot is allowed).

Determine the smallest $k$ for which one can guarantee that there are two checkpoints that are linked by each of the two companies.

*(India)*

**Answer:** $k = n^2 - n + 1$.

**Solution.** We start with showing that for any $k \leqslant n^2 - n$ there may be no pair of checkpoints linked by both companies. Clearly, it suffices to provide such an example for $k = n^2 - n$.

Let company $A$ connect the pairs of checkpoints of the form $(i, i+1)$, where $n \nmid i$. Then all pairs of checkpoints $(i, j)$ linked by $A$ satisfy $\lceil i/n \rceil = \lceil j/n \rceil$.

Let company $B$ connect the pairs of the form $(i, i+n)$, where $1 \leqslant i \leqslant n^2 - n$. Then pairs of checkpoints $(i, j)$ linked by $B$ satisfy $i \equiv j \pmod{n}$. Clearly, no pair $(i, j)$ satisfies both conditions, so there is no pair linked by both companies.

Now we show that for $k = n^2 - n + 1$ there always exist two required checkpoints. Define an *$A$-chain* as a sequence of checkpoints $a_1 < a_2 < \ldots < a_t$ such that company $A$ connects $a_i$ with $a_{i+1}$ for all $1 \leqslant i \leqslant t-1$, but there is no $A$-car transferring from some checkpoint to $a_1$ and no $A$-car transferring from $a_t$ to any other checkpoint. Define *$B$-chains* similarly. Moving forth and back, one easily sees that any checkpoint is included in a unique $A$-chain (possibly consisting of that single checkpoint), as well as in a unique $B$-chain. Now, put each checkpoint into a correspondence to the pair of the $A$-chain and the $B$-chain it belongs to.

All finishing points of $A$-cars are distinct, so there are $n^2 - k = n - 1$ checkpoints that are not such finishing points. Each of them is a starting point of a unique $A$-chain, so the number of $A$-chains is $n-1$. Similarly, the number of $B$-chains also equals $n-1$. Hence, there are $(n-1)^2$ pairs consisting of an $A$- and a $B$-chain. Therefore, two of the $n^2$ checkpoints correspond to the same pair, so that they belong to the same $A$-chain, as well as to the same $B$-chain. This means that they are linked by both companies, as required.

**Comment 1.** The condition that the $i^{\text{th}}$ car starts and finishes lower than the $j^{\text{th}}$ one is used only in the "moving forth and back" argument and in the counting of starting points of the chains. In both cases, the following weaker assumption suffices: No two cars of the same company start at the same checkpoint, and no two such cars finish at the same checkpoint.

Thus, the problem conditions could be weakened in this way,, with no affect on the solution.

**Comment 2.** If the number of checkpoints were $N$, then the answer would be $N - \lceil \sqrt{N} \rceil + 1$. The solution above works verbatim for this generalization.

**C4.** The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 1$. Given an integer $n \geqslant 2$, determine the smallest size of a set $S$ of integers such that for every $k = 2, 3, \ldots, n$ there exist some $x, y \in S$ such that $x - y = F_k$.

*(Croatia)*

**Answer:** $\lceil n/2 \rceil + 1$.

**Solution.** First we show that if a set $S \subset \mathbb{Z}$ satisfies the conditions then $|S| \geqslant \frac{n}{2} + 1$.

Let $d = \lceil n/2 \rceil$, so $n \leqslant 2d \leqslant n + 1$. In order to prove that $|S| \geqslant d + 1$, construct a graph as follows. Let the vertices of the graph be the elements of $S$. For each $1 \leqslant k \leqslant d$, choose two elements $x, y \in S$ such that $x - y = F_{2k-1}$, and add the pair $(x, y)$ to the graph as edge. (Note that by the problem's constraints, there must be a pair $(x, y)$ with $x - y = F_{2k-1}$ for every $3 \leqslant 2k - 1 \leqslant 2d - 1 \leqslant n$; moreover, due to $F_1 = F_2$ we have a pair with $x - y = F_1$ as well.) We will say that the *length* of the edge $(x, y)$ is $|x - y|$.

We claim that the graph contains no cycle. For the sake of contradiction, suppose that the graph contains a cycle $(x_1, \ldots, x_\ell)$, and let the longest edge in the cycle be $(x_1, x_\ell)$ with length $F_{2m+1}$. The other edges $(x_1, x_2), \ldots, (x_{\ell-1}, x_\ell)$ in the cycle are shorter than $F_{2m+1}$ and distinct, their lengths form a subset of $\{F_1, F_3, \ldots, F_{2m-1}\}$. But this is not possible because

$$\begin{aligned} F_{2m+1} = |x_\ell - x_1| &\leqslant \sum_{i=1}^{\ell-1} |x_{i+1} - x_i| \leqslant F_1 + F_3 + F_5 + \ldots + F_{2m-1} \\ &= F_2 + (F_4 - F_2) + (F_6 - F_4) + \ldots + (F_{2m} - F_{2m-2}) = F_{2m} < F_{2m+1}. \end{aligned}$$

Hence, the graph has $d$ edges and cannot contain a cycle, therefore it must contain at least $d + 1$ vertices, so $|S| \geqslant d + 1$.

Now we show a suitable set with $d + 1$ elements. Let

$$S = \{F_0, F_2, F_4, F_5, \ldots, F_{2d}\}.$$

For $1 \leqslant k \leqslant d$ we have $F_0, F_{2k-2}, F_{2k} \in S$ with differences $F_{2k} - F_{2k-2} = F_{2k-1}$ and $F_{2k} - F_0 = F_{2k}$, so each of $F_1, F_2, \ldots, F_{2d}$ occurs as difference between two elements in $S$. So this set containing $d + 1$ numbers is suitable.

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**C5.** Let $p$ be an odd prime, and put $N = \frac{1}{4}(p^3 - p) - 1$. The numbers $1, 2, \ldots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leqslant N$, denote by $r(n)$ the fraction of integers in $\{1, 2, \ldots, n\}$ that are red.

Prove that there exists a positive integer $a \in \{1, 2, \ldots, p-1\}$ such that $r(n) \neq a/p$ for all $n = 1, 2, \ldots, N$.

*(Netherlands)*

**Solution.** Denote by $R(n)$ the number of red numbers in $\{1, 2, \ldots, n\}$, i.e., $R(n) = nr(n)$. Similarly, denote by $B(n)$ and $b(n) = B(n)/n$ the number and proportion of blue numbers in $\{1, 2, \ldots, n\}$, respectively. Notice that $B(n) + R(n) = n$ and $b(n) + r(n) = 1$. Therefore, the statement of the problem does not change after swapping the colors.

Arguing indirectly, for every $a \in \{1, 2, \ldots, p-1\}$ choose some positive integer $n_a$ such that $r(n_a) = a/p$ and, hence, $R(n_a) = an_a/p$. Clearly, $p \mid n_a$, so that $n_a = pm_a$ for some positive integer $m_a$, and $R(n_a) = am_a$. Without loss of generality, we assume that $m_1 < m_{p-1}$, as otherwise one may swap the colors. Notice that

$$m_a \leqslant \frac{N}{p} < \frac{p^2 - 1}{4} \qquad \text{for all } a = 1, 2, \ldots, p-1. \tag{1}$$

The solution is based on a repeated application of the following simple observation.

*Claim.* Assume that $m_a < m_b$ for some $a, b \in \{1, 2, \ldots, p-1\}$. Then

$$m_b \geqslant \frac{a}{b} m_a \qquad \text{and} \qquad m_b \geqslant \frac{p-a}{p-b} m_a.$$

*Proof.* The first inequality follows from $bm_b = R(n_b) \geqslant R(n_a) = am_a$. The second inequality is obtained by swapping colors . □

Let $q = (p-1)/2$. We distinguish two cases.

*Case 1: All $q$ numbers $m_1, m_2, \ldots, m_q$ are smaller than $m_{p-1}$.*

Let $m_a$ be the maximal number among $m_1, m_2, \ldots, m_q$; then $m_a \geqslant q \geqslant a$. Applying the Claim, we get

$$m_{p-1} \geqslant \frac{p-a}{p-(p-1)} m_a \geqslant (p-q)q = \frac{p^2-1}{4},$$

which contradicts (1).

*Case 2: There exists $k \leqslant q$ such that $m_k > m_{p-1}$.*

Choose $k$ to be the smallest index satisfying $m_k > m_{p-1}$; by our assumptions, we have $1 < k \leqslant q < p-1$.

Let $m_a$ be the maximal number among $m_1, m_2, \ldots, m_{k-1}$; then $a \leqslant k-1 \leqslant m_a < m_{p-1}$. Applying the Claim, we get

$$\begin{aligned} m_k \geqslant \frac{p-1}{k} m_{p-1} &\geqslant \frac{p-1}{k} \cdot \frac{p-a}{p-(p-1)} m_a \\ &\geqslant \frac{p-1}{k} \cdot (p-k+1)(k-1) \geqslant \frac{k-1}{k} \cdot (p-1)(p-q) \geqslant \frac{1}{2} \cdot \frac{p^2-1}{2} \end{aligned}$$

which contradicts (1) again.

**Comment 1.** The argument in Case 2, after a slight modification of estimates at the end, applies as soon as there exists $k < \frac{3(p+1)}{4}$ with $a_k < a_{p-1}$. However, this argument does not seem to work if there is no such $k$.

**Comment 2.** If $p$ is small enough, then one can color $\{1, 2, \ldots, N+1\}$ so that there exist numbers $m_1$, $m_2$, …, $m_{p-1}$ satisfying $r(pm_a) = a/p$. For $p = 3, 5, 7$, one can find colorings providing the following sequences:

$$(m_1, m_2) = (1, 2), \qquad (m_1, m_2, m_3, m_4) = (1, 2, 3, 6), \quad \text{and} \quad (m_1, \ldots, , m_6) = (1, 2, 3, 4, 6, 12),$$

respectively.

Thus, for small values of $p$, the number $N$ in the problem statement cannot be increased. However, a careful analysis of the estimates shows that this number can be slightly increased for $p \geqslant 11$.

**C6.** $4n$ coins of weights $1, 2, 3, \ldots, 4n$ are given. Each coin is colored in one of $n$ colors and there are four coins of each color. Show that all these coins can be partitioned into two sets with the same total weight, such that each set contains two coins of each color.

*(Hungary)*

**Solution 1.** Let us pair the coins with weights summing up to $4n + 1$, resulting in the set $S$ of $2n$ pairs: $\{1, 4n\}$, $\{2, 4n - 1\}$, $\ldots$, $\{2n, 2n + 1\}$. It suffices to partition $S$ into two sets, each consisting of $n$ pairs, such that each set contains two coins of each color.

Introduce a multi-graph $G$ (i.e., a graph with loops and multiple edges allowed) on $n$ vertices, so that each vertex corresponds to a color. For each pair of coins from $S$, we add an edge between the vertices corresponding to the colors of those coins. Note that each vertex has degree 4. Also, a desired partition of the coins corresponds to a coloring of the edges of $G$ in two colors, say red and blue, so that each vertex has degree 2 with respect to each color (i.e., each vertex has equal red and blue degrees).

To complete the solution, it suffices to provide such a coloring for each component $G'$ of $G$. Since all degrees of the vertices are even, in $G'$ there exists an Euler circuit $C$ (i.e., a circuit passing through each edge of $G'$ exactly once). Note that the number of edges in $C$ is even (it equals twice the number of vertices in $G'$). Hence all the edges can be colored red and blue so that any two edges adjacent in $C$ have different colors (one may move along $C$ and color the edges one by one alternating red and blue colors). Thus in $G'$ each vertex has equal red and blue degrees, as desired.

**Comment 1.** To complete Solution 1, any partition of the edges of $G$ into circuits of even lengths could be used. In the solution above it was done by the reference to the well-known Euler Circuit Lemma: Let $G$ be a connected graph with all its vertices of even degrees. Then there exists a circuit passing through each edge of $G$ exactly once.

**Solution 2.** As in Solution 1, we will show that it is possible to partition $2n$ pairs $\{1, 4n\}$, $\{2, 4n-1\}$, $\ldots$, $\{2n, 2n+1\}$ into two sets, each consisting of $n$ pairs, such that each set contains two coins of each color.

Introduce a multi-graph (i.e., a graph with multiple edges allowed) $\Gamma$ whose vertices correspond to coins; thus we have $4n$ vertices of $n$ colors so that there are four vertices of each color. Connect pairs of vertices $\{1, 4n\}$, $\{2, 4n - 1\}$, $\ldots$, $\{2n, 2n + 1\}$ by $2n$ *black* edges.

Further, for each monochromatic quadruple of vertices $i, j, k, \ell$ we add a pair of *grey* edges forming a matching, e.g., $(i, j)$ and $(k, \ell)$. In each of $n$ colors of coins we can choose one of three possible matchings; this results in $3^n$ ways of constructing grey edges. Let us call each of $3^n$ possible graphs $\Gamma$ a *cyclic* graph. Note that in a cyclic graph $\Gamma$ each vertex has both black and grey degrees equal to 1. Hence $\Gamma$ is a union of disjoint cycles, and in each cycle black and grey edges alternate (in particular, all cycles have even lengths).

It suffices to find a cyclic graph with all its cycle lengths divisible by 4. Indeed, in this case, for each cycle we start from some vertex, move along the cycle and recolor the black edges either to red or to blue, alternating red and blue colors. Now blue and red edges define the required partition, since for each monochromatic quadruple of vertices the grey edges provide a bijection between the endpoints of red and blue edges.

Among all possible cyclic graphs, let us choose graph $\Gamma_0$ having the minimal number of components (i.e., cycles). The following claim completes the solution.

*Claim.* In $\Gamma_0$, all cycle lengths are divisible by 4.

*Proof.* Assuming the contrary, choose a cycle $C_1$ with an odd number of grey edges. For some color $c$ the cycle $C_1$ contains exactly one grey edge joining two vertices $i$, $j$ of color $c$, while the other edge joining two vertices $k, \ell$ of color $c$ lies in another cycle $C_2$. Now delete edges $(i, j)$ and $(k, \ell)$ and add edges $(i, k)$ and $(j, \ell)$. By this switch we again obtain a cyclic graph $\Gamma_0'$ and decrease the number of cycles by 1. This contradicts the choice of $\Gamma_0$. □

**Comment 2.** Use of an auxiliary graph and reduction to a new problem in terms of this graph is one of the crucial steps in both solutions presented. In fact, graph $G$ from Solution 1 could be obtained from any graph $\Gamma$ from Solution 2 by merging the vertices of the same color.

**C7.** Consider any rectangular table having finitely many rows and columns, with a real number $a(r,c)$ in the cell in row $r$ and column $c$. A pair $(R,C)$, where $R$ is a set of rows and $C$ a set of columns, is called a *saddle pair* if the following two conditions are satisfied:

(*i*) For each row $r'$, there is $r\in R$ such that $a(r,c)\geqslant a(r',c)$ for all $c\in C$;

(*ii*) For each column $c'$, there is $c\in C$ such that $a(r,c)\leqslant a(r,c')$ for all $r\in R$.

A saddle pair $(R,C)$ is called a *minimal pair* if for each saddle pair $(R',C')$ with $R'\subseteq R$ and $C'\subseteq C$, we have $R'=R$ and $C'=C$.

Prove that any two minimal pairs contain the same number of rows.

*(Thailand)*

**Solution 1.** We say that a pair $(R',C')$ of nonempty sets is a *subpair* of a pair $(R,C)$ if $R'\subseteq R$ and $C'\subseteq C$. The subpair is *proper* if at least one of the inclusions is strict.

Let $(R_1,C_1)$ and $(R_2,C_2)$ be two saddle pairs with $|R_1|>|R_2|$. We will find a saddle subpair $(R',C')$ of $(R_1,C_1)$ with $|R'|\leqslant|R_2|$; clearly, this implies the desired statement.

*Step 1: We construct maps $\rho\colon R_1\to R_1$ and $\sigma\colon C_1\to C_1$ such that $|\rho(R_1)|\leqslant|R_2|$, and $a\big(\rho(r_1),c_1\big)\geqslant a\big(r_1,\sigma(c_1)\big)$ for all $r_1\in R_1$ and $c_1\in C_1$.*

Since $(R_1,C_1)$ is a saddle pair, for each $r_2\in R_2$ there is $r_1\in R_1$ such that $a(r_1,c_1)\geqslant a(r_2,c_1)$ for all $c_1\in C_1$; denote one such an $r_1$ by $\rho_1(r_2)$. Similarly, we define four functions

$$\begin{array}{llllll}
\rho_1\colon R_2\to R_1 & \text{such that} & a\big(\rho_1(r_2),c_1\big)\geqslant a(r_2,c_1) & \text{for all} & r_2\in R_2, & c_1\in C_1;\\
\rho_2\colon R_1\to R_2 & \text{such that} & a\big(\rho_2(r_1),c_2\big)\geqslant a(r_1,c_2) & \text{for all} & r_1\in R_1, & c_2\in C_2;\\
\sigma_1\colon C_2\to C_1 & \text{such that} & a\big(r_1,\sigma_1(c_2)\big)\leqslant a(r_1,c_2) & \text{for all} & r_1\in R_1, & c_2\in C_2;\\
\sigma_2\colon C_1\to C_2 & \text{such that} & a\big(r_2,\sigma_2(c_1)\big)\leqslant a(r_2,c_1) & \text{for all} & r_2\in R_2, & c_1\in C_1.
\end{array}\tag{1}$$

Set now $\rho=\rho_1\circ\rho_2\colon R_1\to R_1$ and $\sigma=\sigma_1\circ\sigma_2\colon C_1\to C_1$. We have

$$|\rho(R_1)|=|\rho_1(\rho_2(R_1))|\leqslant|\rho_1(R_2)|\leqslant|R_2|.$$

Moreover, for all $r_1\in R_1$ and $c_1\in C_1$, we get

$$\begin{aligned}a\big(\rho(r_1),c_1\big)=a\big(\rho_1(\rho_2(r_1)),c_1\big)&\geqslant a\big(\rho_2(r_1),c_1\big)\geqslant a\big(\rho_2(r_1),\sigma_2(c_1)\big)\\&\geqslant a\big(r_1,\sigma_2(c_1)\big)\geqslant a\big(r_1,\sigma_1(\sigma_2(c_1))\big)=a\big(r_1,\sigma(c_1)\big),\end{aligned}\tag{2}$$

as desired.

*Step 2: Given maps $\rho$ and $\sigma$, we construct a proper saddle subpair $(R',C')$ of $(R_1,C_1)$.*

The properties of $\rho$ and $\sigma$ yield that

$$a\big(\rho^i(r_1),c_1\big)\geqslant a\big(\rho^{i-1}(r_1),\sigma(c_1)\big)\geqslant\ldots\geqslant a\big(r_1,\sigma^i(c_1)\big),$$

for each positive integer $i$ and all $r_1\in R_1$, $c_1\in C_1$.

Consider the images $R^i=\rho^i(R_1)$ and $C^i=\sigma^i(C_1)$. Clearly, $R_1=R^0\supseteq R^1\supseteq R^2\supseteq\ldots$ and $C_1=C^0\supseteq C^1\supseteq C^2\supseteq\ldots$. Since both chains consist of finite sets, there is an index $n$ such that $R^n=R^{n+1}=\ldots$ and $C^n=C^{n+1}=\ldots$. Then $\rho^n(R^n)=R^{2n}=R^n$, so $\rho^n$ restricted to $R^n$ is a bijection. Similarly, $\sigma^n$ restricted to $C^n$ is a bijection from $C^n$ to itself. Therefore, there exists a positive integer $k$ such that $\rho^{nk}$ acts identically on $R^n$, and $\sigma^{nk}$ acts identically on $C^n$.

We claim now that $(R^n,C^n)$ is a saddle subpair of $(R_1,C_1)$, with $|R^n|\leqslant|R^1|=|\rho(R_1)|\leqslant|R_2|$, which is what we needed. To check that this is a saddle pair, take any row $r'$; since $(R_1,C_1)$ is a saddle pair, there exists $r_1\in R_1$ such that $a(r_1,c_1)\geqslant a(r',c_1)$ for all $c_1\in C_1$. Set now $r_*=\rho^{nk}(r_1)\in R^n$. Then, for each $c\in C^n$ we have $c=\sigma^{nk}(c)$ and hence

$$a(r_*,c)=a\big(\rho^{nk}(r_1),c\big)\geqslant a\big(r_1,\sigma^{nk}(c)\big)=a(r_1,c)\geqslant a(r',c),$$

which establishes condition (*i*). Condition (*ii*) is checked similarly.

**Solution 2.** Denote by $\mathcal{R}$ and $\mathcal{C}$ the set of all rows and the set of all columns of the table, respectively. Let $\mathcal{T}$ denote the given table; for a set $R$ of rows and a set $C$ of columns, let $\mathcal{T}[R, C]$ denote the subtable obtained by intersecting rows from $R$ and columns from $C$.

We say that row $r_1$ *exceeds* row $r_2$ *in range of columns* $C$ (where $C \subseteq \mathcal{C}$) and write $r_1 \succeq_C r_2$ or $r_2 \preceq_C r_1$, if $a(r_1, c) \geqslant a(r_2, c)$ for all $c \in C$. We say that a row $r_1$ is *equal* to a row $r_2$ *in range of columns* $C$ and write $r_1 \equiv_C r_2$, if $a(r_1, c) = a(r_2, c)$ for all $c \in C$. We introduce similar notions, and use the same notation, for columns. Then conditions $(i)$ and $(ii)$ in the definition of a saddle pair can be written as $(i)$ for each $r' \in \mathcal{R}$ there exists $r \in R$ such that $r \succeq_C r'$; and $(ii)$ for each $c' \in \mathcal{C}$ there exists $c \in C$ such that $c \preceq_R c'$.

*Lemma.* Suppose that $(R, C)$ is a minimal pair. Remove from the table several rows outside of $R$ and/or several columns outside of $C$. Then $(R, C)$ remains a minimal pair in the new table.

*Proof.* Obviously, $(R, C)$ remains a saddle pair. Suppose $(R', C')$ is a proper subpair of $(R, C)$. Since $(R, C)$ is a saddle pair, for each row $r^*$ of the initial table, there is a row $r \in R$ such that $r \succeq_C r^*$. If $(R', C')$ became saddle after deleting rows not in $R$ and/or columns not in $C$, there would be a row $r' \in R'$ satisfying $r' \succeq_{C'} r$. Therefore, we would obtain that $r' \succeq_{C'} r^*$, which is exactly condition $(i)$ for the pair $(R', C')$ in the initial table; condition $(ii)$ is checked similarly. Thus, $(R', C')$ was saddle in the initial table, which contradicts the hypothesis that $(R, C)$ was minimal. Hence, $(R, C)$ remains minimal after deleting rows and/or columns. □

By the Lemma, it suffices to prove the statement of the problem in the case $\mathcal{R} = R_1 \cup R_2$ and $\mathcal{C} = C_1 \cup C_2$. Further, suppose that there exist rows that belong both to $R_1$ and $R_2$. Duplicate every such row, and refer one copy of it to the set $R_1$, and the other copy to the set $R_2$. Then $(R_1, C_1)$ and $(R_2, C_2)$ will remain minimal pairs in the new table, with the same numbers of rows and columns, but the sets $R_1$ and $R_2$ will become disjoint. Similarly duplicating columns in $C_1 \cap C_2$, we make $C_1$ and $C_2$ disjoint. Thus it is sufficient to prove the required statement in the case $R_1 \cap R_2 = \varnothing$ and $C_1 \cap C_2 = \varnothing$.

The rest of the solution is devoted to the proof of the following claim including the statement of the problem.

*Claim.* Suppose that $(R_1, C_1)$ and $(R_2, C_2)$ are minimal pairs in table $\mathcal{T}$ such that $R_2 = \mathcal{R} \setminus R_1$ and $C_2 = \mathcal{C} \setminus C_1$. Then $|R_1| = |R_2|$, $|C_1| = |C_2|$; moreover, there are four bijections

$$\begin{aligned}
&\rho_1 \colon R_2 \to R_1 \quad \text{such that} \quad \rho_1(r_2) \equiv_{C_1} r_2 \quad \text{for all} \quad r_2 \in R_2;\\
&\rho_2 \colon R_1 \to R_2 \quad \text{such that} \quad \rho_2(r_1) \equiv_{C_2} r_1 \quad \text{for all} \quad r_1 \in R_1;\\
&\sigma_1 \colon C_2 \to C_1 \quad \text{such that} \quad \sigma_1(c_2) \equiv_{R_1} c_2 \quad \text{for all} \quad c_2 \in C_2;\\
&\sigma_2 \colon C_1 \to C_2 \quad \text{such that} \quad \sigma_2(c_1) \equiv_{R_2} c_1 \quad \text{for all} \quad c_1 \in C_1.
\end{aligned} \tag{3}$$

We prove the Claim by induction on $|\mathcal{R}| + |\mathcal{C}|$. In the base case we have $|R_1| = |R_2| = |C_1| = |C_2| = 1$; let $R_i = \{r_i\}$ and $C_i = \{c_i\}$. Since $(R_1, C_1)$ and $(R_2, C_2)$ are saddle pairs, we have $a(r_1, c_1) \geqslant a(r_2, c_1) \geqslant a(r_2, c_2) \geqslant a(r_1, c_2) \geqslant a(r_1, c_1)$, hence, the table consists of four equal numbers, and the statement follows.

To prove the inductive step, introduce the maps $\rho_1$, $\rho_2$, $\sigma_1$, and $\sigma_2$ as in Solution 1, see (1). Suppose first that all four maps are surjective. Then, in fact, we have $|R_1| = |R_2|$, $|C_1| = |C_2|$, and all maps are bijective. Moreover, for all $r_2 \in R_2$ and $c_2 \in C_2$ we have

$$\begin{aligned}
a(r_2, c_2) \leqslant a\big(r_2, \sigma_2^{-1}(c_2)\big) \leqslant a\big(\rho_1(r_2), \sigma_2^{-1}(c_2)\big) \leqslant a\big(\rho_1(r_2), \sigma_1^{-1} \circ \sigma_2^{-1}(c_2)\big)\\
\leqslant a\big(\rho_2 \circ \rho_1(r_2), \sigma_1^{-1} \circ \sigma_2^{-1}(c_2)\big).
\end{aligned} \tag{4}$$

Summing up, we get

$$\sum_{\substack{r_2 \in R_2\\ c_2 \in C_2}} a(r_2, c_2) \leqslant \sum_{\substack{r_2 \in R_2\\ c_2 \in C_2}} a\big(\rho_2 \circ \rho_1(r_2), \sigma_1^{-1} \circ \sigma_2^{-1}(c_2)\big).$$

Since $\rho_1 \circ \rho_2$ and $\sigma_1^{-1} \circ \sigma_2^{-1}$ are permutations of $R_2$ and $C_2$, respectively, this inequality is in fact equality. Therefore, all inequalities in (4) turn into equalities, which establishes the inductive step in this case.

It remains to show that all four maps are surjective. For the sake of contradiction, we assume that $\rho_1$ is not surjective. Now let $R_1' = \rho_1(R_2)$ and $C_1' = \sigma_1(C_2)$, and set $R^* = R_1 \setminus R_1'$ and $C^* = C_1 \setminus C_1'$. By our assumption, $R^* \neq \varnothing$.

Let $\mathcal{Q}$ be the table obtained from $\mathcal{T}$ by removing the rows in $R^*$ and the columns in $C^*$; in other words, $\mathcal{Q} = \mathcal{T}[R_1' \cup R_2, C_1' \cup C_2]$. By the definition of $\rho_1$, for each $r_2 \in R_2$ we have $\rho_1(r_2) \geq_{C_1} r_2$, so a fortiori $\rho_1(r_2) \geq_{C_1'} r_2$; moreover, $\rho_1(r_2) \in R_1'$. Similarly, $C_1' \ni \sigma_1(c_2) \leq_{R_1'} c_2$ for each $c_2 \in C_2$. This means that $(R_1', C_1')$ is a saddle pair in $\mathcal{Q}$. Recall that $(R_2, C_2)$ remains a minimal pair in $\mathcal{Q}$, due to the Lemma.

Therefore, $\mathcal{Q}$ admits a minimal pair $(\overline{R}_1, \overline{C}_1)$ such that $\overline{R}_1 \subseteq R_1'$ and $\overline{C}_1 \subseteq C_1'$. For a minute, confine ourselves to the subtable $\overline{\mathcal{Q}} = \mathcal{Q}[\overline{R}_1 \cup R_2, \overline{C}_1 \cup C_2]$. By the Lemma, the pairs $(\overline{R}_1, \overline{C}_1)$ and $(R_2, C_2)$ are also minimal in $\overline{\mathcal{Q}}$. By the inductive hypothesis, we have $|R_2| = |\overline{R}_1| \leqslant |R_1'| = |\rho_1(R_2)| \leqslant |R_2|$, so all these inequalities are in fact equalities. This implies that $\overline{R}_2 = R_2'$ and that $\rho_1$ is a bijection $R_2 \to R_1'$. Similarly, $\overline{C}_1 = C_1'$, and $\sigma_1$ is a bijection $C_2 \to C_1'$. In particular, $(R_1', C_1')$ is a minimal pair in $\mathcal{Q}$.

Now, by inductive hypothesis again, we have $|R_1'| = |R_2|$, $|C_1'| = |C_2|$, and there exist four bijections

$$\begin{aligned}
&\rho_1' \colon R_2 \to R_1' \quad \text{such that} \quad \rho_1'(r_2) \equiv_{C_1'} r_2 \quad \text{for all} \quad r_2 \in R_2;\\
&\rho_2' \colon R_1' \to R_2 \quad \text{such that} \quad \rho_2'(r_1) \equiv_{C_2} r_1 \quad \text{for all} \quad r_1 \in R_1';\\
&\sigma_1' \colon C_2 \to C_1' \quad \text{such that} \quad \sigma_1'(c_2) \equiv_{R_1'} c_2 \quad \text{for all} \quad c_2 \in C_2;\\
&\sigma_2' \colon C_1' \to C_2 \quad \text{such that} \quad \sigma_2'(c_1) \equiv_{R_2} c_1 \quad \text{for all} \quad c_1 \in C_1'.
\end{aligned}$$

Notice here that $\sigma_1$ and $\sigma_1'$ are two bijections $C_2 \to C_1'$ satisfying $\sigma_1'(c_2) \equiv_{R_1'} c_2 \geq_{R_1} \sigma_1(c_2)$ for all $c_2 \in C_2$. Now, if $\sigma_1'(c_2) \neq \sigma_1(c_2)$ for some $c_2 \in C_2$, then we could remove column $\sigma_1'(c_2)$ from $C_1'$ obtaining another saddle pair $\big(R_1', C_1' \setminus \{\sigma_1'(c_2)\}\big)$ in $\mathcal{Q}$. This is impossible for a minimal pair $(R_1', C_1')$; hence the maps $\sigma_1$ and $\sigma_1'$ coincide.

Now we are prepared to show that $(R_1', C_1')$ is a saddle pair in $\mathcal{T}$, which yields a desired contradiction (since $(R_1, C_1)$ is not minimal). By symmetry, it suffices to find, for each $r' \in \mathcal{R}$, a row $r_1 \in R_1'$ such that $r_1 \geq_{C_1'} r'$. If $r' \in R_2$, then we may put $r_1 = \rho_1(r')$; so, in the sequel we assume $r' \in R_1$.

There exists $r_2 \in R_2$ such that $r' \leq_{C_2} r_2$; set $r_1 = (\rho_2')^{-1}(r_2) \in R_1'$ and recall that $r_1 \equiv_{C_2} r_2 \geq_{C_2} r'$. Therefore, implementing the bijection $\sigma_1 = \sigma_1'$, for each $c_1 \in C_1'$ we get

$$a(r', c_1) \leqslant a\big(r', \sigma_1^{-1}(c_1)\big) \leqslant a\big(r_1, \sigma_1^{-1}(c_1)\big) = a\big(r_1, \sigma_1' \circ \sigma_1^{-1}(c_1)\big) = a\big(r_1, c_1\big),$$

which shows $r' \leq_{C_1'} r_1$, as desired. The inductive step is completed.

**Comment 1.** For two minimal pairs $(R_1, C_1)$ and $(R_2, C_2)$, Solution 2 not only proves the required equalities $|R_1| = |R_2|$ and $|C_1| = |C_2|$, but also shows the existence of bijections (3). In simple words, this means that the four subtables $\mathcal{T}[R_1, C_1]$, $\mathcal{T}[R_1, C_2]$, $\mathcal{T}[R_2, C_1]$, and $\mathcal{T}[R_2, C_2]$ differ only by permuting rows/columns. Notice that the existence of such bijections immediately implies that $(R_1, C_2)$ and $(R_2, C_1)$ are also minimal pairs.

This stronger claim may also be derived directly from the arguments in Solution 1, even without the assumptions $R_1 \cap R_2 = \varnothing$ and $C_1 \cap C_2 = \varnothing$. Indeed, if $|R_1| = |R_2|$ and $|C_1| = |C_2|$, then similar arguments show that $R^n = R_1$, $C^n = C_1$, and for any $r \in R^n$ and $c \in C^n$ we have

$$a(r, c) = a\big(\rho^{nk}(r), c\big) \geqslant a\big(\rho^{nk-1}(r), \sigma(c)\big) \geqslant \ldots \geqslant a\big(r, \sigma^{nk}(c)\big) = a(r, c).$$

This yields that all above inequalities turn into equalities. Moreover, this yields that all inequalities in (2) turn into equalities. Hence $\rho_1$, $\rho_2$, $\sigma_1$, and $\sigma_2$ satisfy (3).

It is perhaps worth mentioning that one cannot necessarily find the maps in (3) so as to satisfy $\rho_1 = \rho_2^{-1}$ and $\sigma_1 = \sigma_2^{-1}$, as shown by the table below.

| | | | |
|---|---|---|---|
| 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 |

**Comment 2.** One may use the following, a bit more entertaining formulation of the same problem.

On a specialized market, a finite number of products are being sold, and there are finitely many retailers each selling all the products by some prices. Say that retailer $r_1$ *dominates* retailer $r_2$ with respect to a set of products $P$ if $r_1$'s price of each $p \in P$ does not exceed $r_2$'s price of $p$. Similarly, product $p_1$ *exceeds* product $p_2$ with respect to a set of retailers $R$, if $r$'s price of $p_1$ is not less than $r$'s price of $p_2$, for each $r \in R$.

Say that a set $R$ of retailers and a set $P$ of products form a *saddle pair* if for each retailer $r'$ there is $r \in R$ dominating $r'$ with respect to $P$, and for each product $p'$ there is $p \in P$ exceeding $p'$ with respect to $R$. A saddle pair $(R, P)$ is called a *minimal pair* if for each saddle pair $(R', P')$ with $R' \subseteq R$ and $P' \subseteq P$, we have $R' = R$ and $P' = P$.

Prove that any two minimal pairs contain the same number of retailers.

**C8.** Players $A$ and $B$ play a game on a blackboard that initially contains 2020 copies of the number 1. In every round, player $A$ erases two numbers $x$ and $y$ from the blackboard, and then player $B$ writes one of the numbers $x+y$ and $|x-y|$ on the blackboard. The game terminates as soon as, at the end of some round, one of the following holds:

(1) one of the numbers on the blackboard is larger than the sum of all other numbers;

(2) there are only zeros on the blackboard.

Player $B$ must then give as many cookies to player $A$ as there are numbers on the blackboard. Player $A$ wants to get as many cookies as possible, whereas player $B$ wants to give as few as possible. Determine the number of cookies that $A$ receives if both players play optimally.

*(Austria)*

**Answer:** 7.

**Solution.** For a positive integer $n$, we denote by $S_2(n)$ the sum of digits in its binary representation. We prove that, in fact, if a board initially contains an even number $n > 1$ of ones, then $A$ can guarantee to obtain $S_2(n)$, but not more, cookies. The binary representation of 2020 is $2020 = \overline{11111100100}_2$, so $S_2(2020) = 7$, and the answer follows.

**A strategy for $A$.** *At any round, while possible, $A$ chooses two equal nonzero numbers on the board.* Clearly, while $A$ can make such choice, the game does not terminate. On the other hand, $A$ can follow this strategy unless the game has already terminated. Indeed, if $A$ always chooses two equal numbers, then each number appearing on the board is either 0 or a power of 2 with non-negative integer exponent, this can be easily proved using induction on the number of rounds. At the moment when $A$ is unable to follow the strategy all nonzero numbers on the board are distinct powers of 2. If the board contains at least one such power, then the largest of those powers is greater than the sum of the others. Otherwise there are only zeros on the blackboard, in both cases the game terminates.

For every number on the board, define its *range* to be the number of ones it is obtained from. We can prove by induction on the number of rounds that for any nonzero number $k$ written by $B$ its range is $k$, and for any zero written by $B$ its range is a power of 2. Thus at the end of each round all the ranges are powers of two, and their sum is $n$. Since $S_2(a+b) \leqslant S_2(a) + S_2(b)$ for any positive integers $a$ and $b$, the number $n$ cannot be represented as a sum of less than $S_2(n)$ powers of 2. Thus at the end of each round the board contains at least $S_2(n)$ numbers, while $A$ follows the above strategy. So $A$ can guarantee at least $S_2(n)$ cookies for himself.

**Comment.** There are different proofs of the fact that the presented strategy guarantees at least $S_2(n)$ cookies for $A$. For instance, one may denote by $\Sigma$ the sum of numbers on the board, and by $z$ the number of zeros. Then the board contains at least $S_2(\Sigma) + z$ numbers; on the other hand, during the game, the number $S_2(\Sigma) + z$ does not decrease, and its initial value is $S_2(n)$. The claim follows.

**A strategy for $B$.** Denote $s = S_2(n)$.

Let $x_1, \ldots, x_k$ be the numbers on the board at some moment of the game after $B$'s turn or at the beginning of the game. Say that a collection of $k$ signs $\varepsilon_1, \ldots, \varepsilon_k \in \{+1, -1\}$ is *balanced* if

$$\sum_{i=1}^{k} \varepsilon_i x_i = 0.$$

We say that a situation on the board is *good* if $2^{s+1}$ does not divide the number of balanced collections. An appropriate strategy for $B$ can be explained as follows: *Perform a move so that the situation remains good, while it is possible.* We intend to show that in this case $B$ will not lose more than $S_2(n)$ cookies. For this purpose, we prove several lemmas.

For a positive integer $k$, denote by $\nu_2(k)$ the exponent of the largest power of 2 that divides $k$. Recall that, by Legendre's formula, $\nu_2(n!) = n - S_2(n)$ for every positive integer $n$.

*Lemma 1.* The initial situation is good.

*Proof.* In the initial configuration, the number of balanced collections is equal to $\binom{n}{n/2}$. We have

$$\nu_2\left(\binom{n}{n/2}\right) = \nu_2(n!) - 2\nu_2\big((n/2)!\big) = \big(n - S_2(n)\big) - 2\left(\frac{n}{2} - S_2(n/2)\right) = S_2(n) = s.$$

Hence $2^{s+1}$ does not divide the number of balanced collections, as desired. □

*Lemma 2.* $B$ may play so that after each round the situation remains good.

*Proof.* Assume that the situation $(x_1, \ldots, x_k)$ before a round is good, and that $A$ erases two numbers, $x_p$ and $x_q$.

Let $N$ be the number of all balanced collections, $N_+$ be the number of those having $\varepsilon_p = \varepsilon_q$, and $N_-$ be the number of other balanced collections. Then $N = N_+ + N_-$. Now, if $B$ replaces $x_p$ and $x_q$ by $x_p + x_q$, then the number of balanced collections will become $N_+$. If $B$ replaces $x_p$ and $x_q$ by $|x_p - x_q|$, then this number will become $N_-$. Since $2^{s+1}$ does not divide $N$, it does not divide one of the summands $N_+$ and $N_-$, hence $B$ can reach a good situation after the round. □

*Lemma 3.* Assume that the game terminates at a good situation. Then the board contains at most $s$ numbers.

*Proof.* Suppose, one of the numbers is greater than the sum of the other numbers. Then the number of balanced collections is 0 and hence divisible by $2^{s+1}$. Therefore, the situation is not good.

Then we have only zeros on the blackboard at the moment when the game terminates. If there are $k$ of them, then the number of balanced collections is $2^k$. Since the situation is good, we have $k \leqslant s$. □

By Lemmas 1 and 2, $B$ may act in such way that they keep the situation good. By Lemma 3, when the game terminates, the board contains at most $s$ numbers. This is what we aimed to prove.

**Comment 1.** If the initial situation had some odd number $n > 1$ of ones on the blackboard, player $A$ would still get $S_2(n)$ cookies, provided that both players act optimally. The proof of this fact is similar to the solution above, after one makes some changes in the definitions. Such changes are listed below.

Say that a collection of $k$ signs $\varepsilon_1, \ldots, \varepsilon_k \in \{+1, -1\}$ is *positive* if

$$\sum_{i=1}^{k} \varepsilon_i x_i > 0.$$

For every index $i = 1, 2, \ldots, k$, we denote by $N_i$ the number of positive collections such that $\varepsilon_i = 1$. Finally, say that a situation on the board is *good* if $2^{s-1}$ does not divide at least one of the numbers $N_i$. Now, a strategy for $B$ again consists in preserving the situation good, after each round.

**Comment 2.** There is an easier strategy for $B$, allowing, in the game starting with an even number $n$ of ones, to lose no more than $d = \lfloor \log_2(n+2) \rfloor - 1$ cookies. If the binary representation of $n$ contains at most two zeros, then $d = S_2(n)$, and hence the strategy is optimal in that case. We outline this approach below.

First of all, we can assume that $A$ never erases zeros from the blackboard. Indeed, $A$ may skip such moves harmlessly, ignoring the zeros in the further process; this way, $A$'s win will just increase.

We say that a situation on the blackboard is *pretty* if the numbers on the board can be partitioned into two groups with equal sums. Clearly, if the situation before some round is pretty, then $B$ may play so as to preserve this property after the round. The strategy for $B$ is as follows:

• $B$ always chooses a move that leads to a pretty situation.

• If both possible moves of $B$ lead to pretty situations, then $B$ writes the sum of the two numbers erased by $A$.

Since the situation always remains pretty, the game terminates when all numbers on the board are zeros.

Suppose that, at the end of the game, there are $m \geqslant d+1 = \lfloor \log_2(n+2) \rfloor$ zeros on the board; then $2^m - 1 > n/2$.

Now we analyze the whole process of the play. Let us number the zeros on the board in order of appearance. During the play, each zero had appeared after subtracting two equal numbers. Let $n_1, \ldots, n_m$ be positive integers such that the first zero appeared after subtracting $n_1$ from $n_1$, the second zero appeared after subtracting $n_2$ from $n_2$, and so on. Since the sum of the numbers on the blackboard never increases, we have $2n_1 + \ldots + 2n_m \leqslant n$, hence

$$n_1 + \ldots + n_m \leqslant n/2 < 2^m - 1.$$

There are $2^m$ subsets of the set $\{1, 2, \ldots, m\}$. For $I \subseteq \{1, 2, \ldots, m\}$, denote by $f(I)$ the sum $\sum_{i \in I} n_i$. There are less than $2^m$ possible values for $f(I)$, so there are two distinct subsets $I$ and $J$ with $f(I) = f(J)$. Replacing $I$ and $J$ with $I \setminus J$ and $J \setminus I$, we assume that $I$ and $J$ are disjoint.

Let $i_0$ be the smallest number in $I \cup J$; without loss of generality, $i_0 \in I$. Consider the round when $A$ had erased two numbers equal to $n_{i_0}$, and $B$ had put the $i_0{}^{\text{th}}$ zero instead, and the situation *before* that round.

For each nonzero number $z$ which is on the blackboard at this moment, we can keep track of it during the further play until it enters one of the numbers $n_i$, $i \geqslant i_0$, which then turn into zeros. For every $i = i_0, i_0 + 1, \ldots, m$, we denote by $X_i$ the collection of all numbers on the blackboard that finally enter the first copy of $n_i$, and by $Y_i$ the collection of those finally entering the second copy of $n_i$. Thus, each of $X_{i_0}$ and $Y_{i_0}$ consists of a single number. Since $A$ never erases zeros, the $2(m - i_0 + 1)$ defined sets are pairwise disjoint.

Clearly, for either of the collections $X_i$ and $Y_i$, a signed sum of its elements equals $n_i$, for a proper choice of the signs. Therefore, for every $i = i_0, i_0 + 1, \ldots, m$ one can endow numbers in $X_i \cup Y_i$ with signs so that their sum becomes any of the numbers $-2n_i$, 0, or $2n_i$. Perform such endowment so as to get $2n_i$ from each collection $X_i \cup Y_i$ with $i \in I$, $-2n_j$ from each collection $X_j \cup Y_j$ with $j \in J$, and 0 from each remaining collection. The obtained signed sum of all numbers on the blackboard equals

$$\sum_{i \in I} 2n_i - \sum_{i \in J} 2n_i = 0,$$

and the numbers in $X_{i_0}$ and $Y_{i_0}$ have the same (positive) sign.

This means that, at this round, $B$ could add up the two numbers $n_{i_0}$ to get a pretty situation. According to the strategy, $B$ should have performed that, instead of subtracting the numbers. This contradiction shows that $m \leqslant d$, as desired.

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# Geometry

**G1.** Let $ABC$ be an isosceles triangle with $BC = CA$, and let $D$ be a point inside side $AB$ such that $AD < DB$. Let $P$ and $Q$ be two points inside sides $BC$ and $CA$, respectively, such that $\angle DPB = \angle DQA = 90^\circ$. Let the perpendicular bisector of $PQ$ meet line segment $CQ$ at $E$, and let the circumcircles of triangles $ABC$ and $CPQ$ meet again at point $F$, different from $C$.

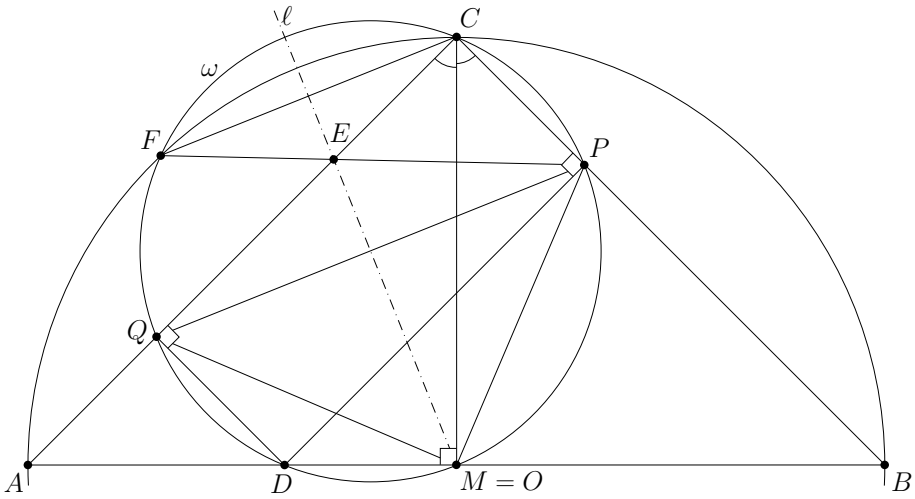
Suppose that $P, E, F$ are collinear. Prove that $\angle ACB = 90^\circ$.

*(Luxembourg)*

**Solution 1.** Let $\ell$ be the perpendicular bisector of $PQ$, and denote by $\omega$ the circle $CFPQ$. By $DP \perp BC$ and $DQ \perp AC$, the circle $\omega$ passes through $D$; moreover, $CD$ is a diameter of $\omega$.

The lines $QE$ and $PE$ are symmetric about $\ell$, and $\ell$ is a symmetry axis of $\omega$ as well; it follows that the chords $CQ$ and $FP$ are symmetric about $\ell$, hence $C$ and $F$ are symmetric about $\ell$. Therefore, the perpendicular bisector of $CF$ coincides with $\ell$. Thus $\ell$ passes through the circumcenter $O$ of $ABC$.

Let $M$ be the midpoint of $AB$. Since $CM \perp DM$, $M$ also lies on $\omega$. By $\angle ACM = \angle BCM$, the chords $MP$ and $MQ$ of $\omega$ are equal. Then, from $MP = MQ$ it follows that $\ell$ passes through $M$.




Finally, both $O$ and $M$ lie on lines $\ell$ and $CM$, therefore $O = M$, and $\angle ACB = 90^\circ$ follows.

**Solution 2.** Like in the first solution, we conclude that points $C$, $P$, $Q$, $D$, $F$ and the midpoint $M$ of $AB$ lie on one circle $\omega$ with diameter $CD$, and $M$ lies on $\ell$, the perpendicular bisector of $PQ$.

Let $BF$ and $CM$ meet at $G$ and let $\alpha = \angle ABF$. Then, since $E$ lies on $\ell$, and the quadrilaterals $FCBA$ and $FCPQ$ are cyclic, we have

$$\angle CQP = \angle FPQ = \angle FCQ = \angle FCA = \angle FBA = \alpha.$$

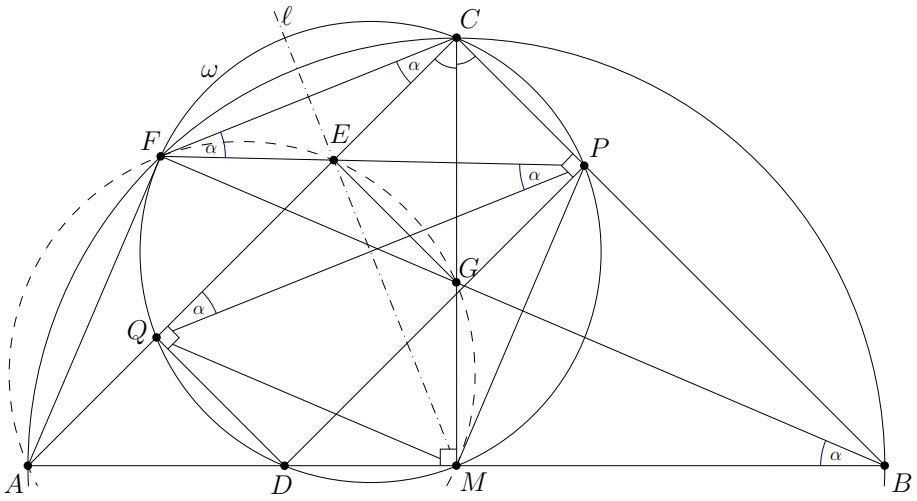
Since points $P$, $E$, $F$ are collinear, we have

$$\angle FEM = \angle FEQ + \angle QEM = 2\alpha + (90^\circ - \alpha) = 90^\circ + \alpha.$$

But $\angle FGM = 90^\circ + \alpha$, so $FEGM$ is cyclic. Hence

$$\angle EGC = \angle EFM = \angle PFM = \angle PCM.$$

Thus $GE \parallel BC$. It follows that $\angle FAC = \angle CBF = \angle EGF$, so $FEGA$ is cyclic, too. Hence $\angle ACB = \angle AFB = \angle AFG = 180^\circ - \angle AMG = 90^\circ$, that completes the proof.




**Comment 1.** The converse statement is true: if $\angle ACB = 90^\circ$ then points $P$, $E$ and $F$ are collinear. This direction is easier to prove.

**Comment 2.** The statement of the problem remains true if the projection $P$ of $D$ onto $BC$ lies outside line segment $BC$. The restriction that $P$ lies inside line segment $BC$ is given to reduce case-sensitivity.

**G2.** Let $ABCD$ be a convex quadrilateral. Suppose that $P$ is a point in the interior of $ABCD$ such that

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

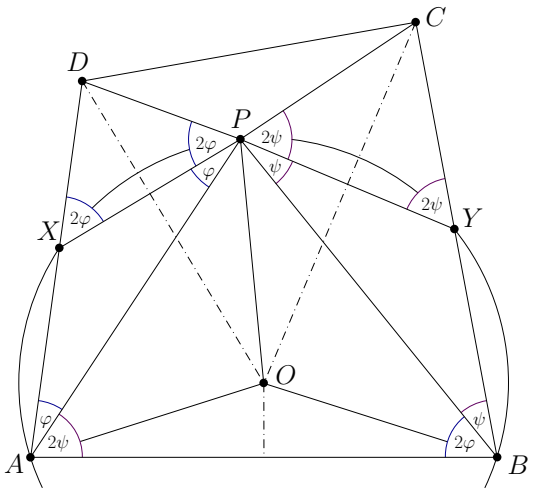
The internal bisectors of angles $ADP$ and $PCB$ meet at a point $Q$ inside the triangle $ABP$. Prove that $AQ = BQ$.

*(Poland)*

**Solution 1.** Let $\varphi = \angle PAD$ and $\psi = \angle CBP$; then we have $\angle PBA = 2\varphi$, $\angle DPA = 3\varphi$, $\angle BAP = 2\psi$ and $\angle BPC = 3\psi$. Let $X$ be the point on segment $AD$ with $\angle XPA = \varphi$. Then

$$\angle PXD = \angle PAX + \angle XPA = 2\varphi = \angle DPA - \angle XPA = \angle DPX.$$

It follows that triangle $DPX$ is isosceles with $DX = DP$ and therefore the internal angle bisector of $\angle ADP$ coincides with the perpendicular bisector of $XP$. Similarly, if $Y$ is a point on $BC$ such that $\angle BPY = \psi$, then the internal angle bisector of $\angle PCB$ coincides with the perpendicular bisector of $PY$. Hence, we have to prove that the perpendicular bisectors of $XP$, $PY$, and $AB$ are concurrent.




Notice that

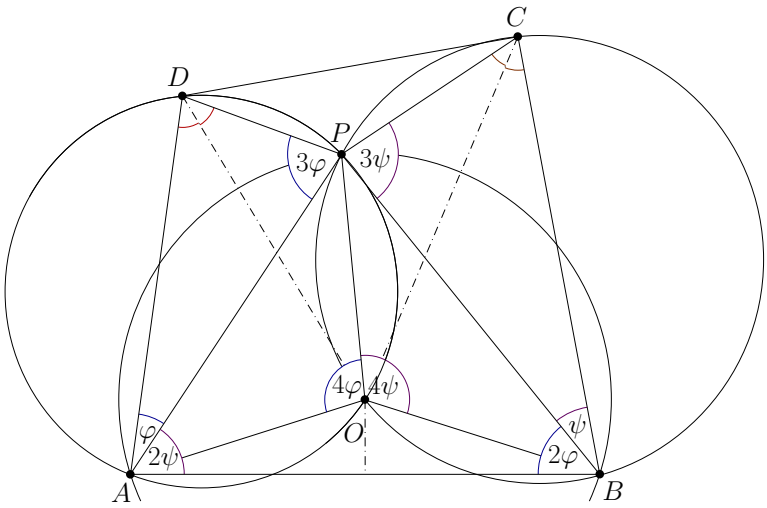
$$\angle AXP = 180^\circ - \angle PXD = 180^\circ - 2\varphi = 180^\circ - \angle PBA.$$

Hence the quadrilateral $AXPB$ is cyclic; in other words, $X$ lies on the circumcircle of triangle $APB$. Similarly, $Y$ lies on the circumcircle of triangle $APB$. It follows that the perpendicular bisectors of $XP$, $PY$, and $AB$ all pass through the center of circle $(ABYPX)$. This finishes the proof.

**Comment.** Introduction of points $X$ and $Y$ seems to be the key step in the solution above. Note that the same point $X$ could be introduced in different ways, e.g., as the point on the ray $CP$ beyond $P$ such that $\angle PBX = \varphi$, or as a point where the circle $(APB)$ meets again $AB$. Different definitions of $X$ could lead to different versions of the further solution.

**Solution 2.** We define the angles $\varphi = \angle PAD$, $\psi = \angle CBP$ and use $\angle PBA = 2\varphi$, $\angle DPA = 3\varphi$, $\angle BAP = 2\psi$ and $\angle BPC = 3\psi$ again. Let $O$ be the circumcenter of $\triangle APB$.

Notice that $\angle ADP = 180^\circ - \angle PAD - \angle DPA = 180^\circ - 4\varphi$, which, in particular, means that $4\varphi < 180^\circ$. Further, $\angle POA = 2\angle PBA = 4\varphi = 180^\circ - \angle ADP$, therefore the quadrilateral $ADPO$ is cyclic. By $AO = OP$, it follows that $\angle ADO = \angle ODP$. Thus $DO$ is the internal bisector of $\angle ADP$. Similarly, $CO$ is the internal bisector of $\angle PCB$.



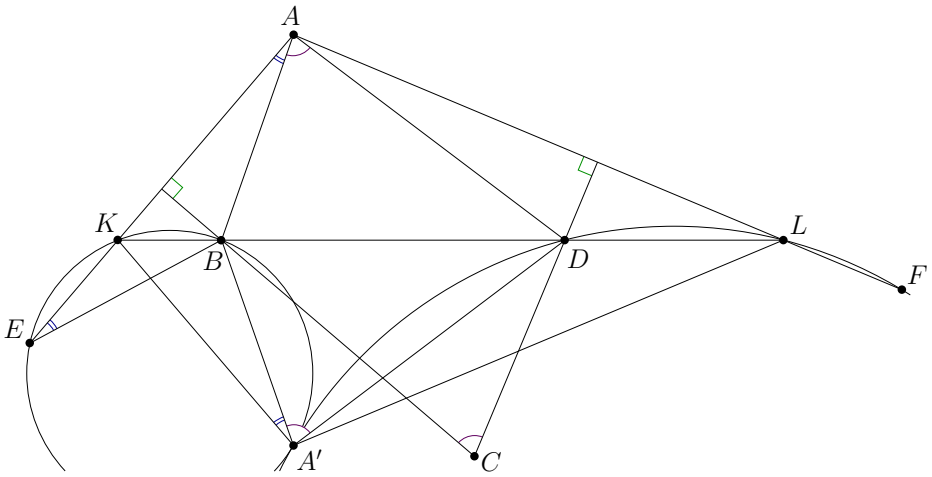

Finally, $O$ lies on the perpendicular bisector of $AB$ as it is the circumcenter of $\triangle APB$. Therefore the three given lines in the problem statement concur at point $O$.

**G3.** Let $ABCD$ be a convex quadrilateral with $\angle ABC > 90^\circ$, $\angle CDA > 90^\circ$, and $\angle DAB = \angle BCD$. Denote by $E$ and $F$ the reflections of $A$ in lines $BC$ and $CD$, respectively. Suppose that the segments $AE$ and $AF$ meet the line $BD$ at $K$ and $L$, respectively. Prove that the circumcircles of triangles $BEK$ and $DFL$ are tangent to each other.

*(Slovakia)*

**Solution 1.** Denote by $A'$ the reflection of $A$ in $BD$. We will show that that the quadrilaterals $A'BKE$ and $A'DLF$ are cyclic, and their circumcircles are tangent to each other at point $A'$.

From the symmetry about line $BC$ we have $\angle BEK = \angle BAK$, while from the symmetry in $BD$ we have $\angle BAK = \angle BA'K$. Hence $\angle BEK = \angle BA'K$, which implies that the quadrilateral $A'BKE$ is cyclic. Similarly, the quadrilateral $A'DLF$ is also cyclic.




For showing that circles $A'BKE$ and $A'DLF$ are tangent it suffices to prove that

$$\angle A'KB + \angle A'LD = \angle BA'D.$$

Indeed, by $AK \perp BC$, $AL \perp CD$, and again the symmetry in $BD$ we have

$$\angle A'KB + \angle A'LD = 180^\circ - \angle KA'L = 180^\circ - \angle KAL = \angle BCD = \angle BAD = \angle BA'D,$$

as required.

**Comment 1.** The key to the solution above is introducing the point $A'$; then the angle calculations can be done in many different ways.

**Solution 2.** Note that $\angle KAL = 180^\circ - \angle BCD$, since $AK$ and $AL$ are perpendicular to $BC$ and $CD$, respectively. Reflect both circles $(BEK)$ and $(DFL)$ in $BD$. Since $\angle KEB = \angle KAB$ and $\angle DFL = \angle DAL$, the images are the circles $(KAB)$ and $(LAD)$, respectively; so they meet at $A$. We need to prove that those two reflections are tangent at $A$.

For this purpose, we observe that

$$\angle AKB + \angle ALD = 180^\circ - \angle KAL = \angle BCD = \angle BAD.$$

Thus, there exists a ray $AP$ inside angle $\angle BAD$ such that $\angle BAP = \angle AKB$ and $\angle DAP = \angle DLA$. Hence the line $AP$ is a common tangent to the circles $(KAB)$ and $(LAD)$, as desired.

**Comment 2.** The statement of the problem remains true for a more general configuration, e.g., if line $BD$ intersect the extension of segment $AE$ instead of the segment itself, etc. The corresponding restrictions in the statement are given to reduce case sensitivity.

**G4.** In the plane, there are $n \geqslant 6$ pairwise disjoint disks $D_1, D_2, \ldots, D_n$ with radii $R_1 \geqslant R_2 \geqslant \ldots \geqslant R_n$. For every $i = 1, 2, \ldots, n$, a point $P_i$ is chosen in disk $D_i$. Let $O$ be an arbitrary point in the plane. Prove that

$$OP_1 + OP_2 + \ldots + OP_n \geqslant R_6 + R_7 + \ldots + R_n.$$

(A disk is assumed to contain its boundary.)

*(Iran)*

**Solution.** We will make use of the following lemma.

*Lemma.* Let $D_1, \ldots, D_6$ be disjoint disks in the plane with radii $R_1, \ldots, R_6$. Let $P_i$ be a point in $D_i$, and let $O$ be an arbitrary point. Then there exist indices $i$ and $j$ such that $OP_i \geqslant R_j$.

*Proof.* Let $O_i$ be the center of $D_i$. Consider six rays $OO_1$, ..., $OO_6$ (if $O = O_i$, then the ray $OO_i$ may be assumed to have an arbitrary direction). These rays partition the plane into six angles (one of which may be non-convex) whose measures sum up to $360^\circ$; hence one of the angles, say $\angle O_iOO_j$, has measure at most $60^\circ$. Then $O_iO_j$ cannot be the unique largest side in (possibly degenerate) triangle $OO_iO_j$, so, without loss of generality, $OO_i \geqslant O_iO_j \geqslant R_i + R_j$. Therefore, $OP_i \geqslant OO_i - R_i \geqslant (R_i + R_j) - R_i = R_j$, as desired. □

Now we prove the required inequality by induction on $n \geqslant 5$. The base case $n = 5$ is trivial. For the inductive step, apply the Lemma to the six largest disks, in order to find indices $i$ and $j$ such that $1 \leqslant i, j \leqslant 6$ and $OP_i \geqslant R_j \geqslant R_6$. Removing $D_i$ from the configuration and applying the inductive hypothesis, we get

$$\sum_{k \neq i} OP_k \geqslant \sum_{\ell \geqslant 7} R_\ell.$$

Adding up this inequality with $OP_i \geqslant R_6$ we establish the inductive step.

**Comment 1.** It is irrelevant to the problem whether the disks contain their boundaries or not. This condition is included for clarity reasons only. The problem statement remains true, and the solution works verbatim, if the disks are assumed to have disjoint interiors.

**Comment 2.** There are several variations of the above solution. In particular, while performing the inductive step, one may remove the disk with the largest value of $OP_i$ and apply the inductive hypothesis to the remaining disks (the Lemma should still be applied to the six largest disks).

**Comment 3.** While proving the Lemma, one may reduce it to a particular case when the disks are congruent, as follows: Choose the smallest radius $r$ of the disks in the Lemma statement, and then replace, for each $i$, the $i^{\text{th}}$ disk with its homothetic copy, using the homothety centered at $P_i$ with ratio $r/R_i$.

This argument shows that the Lemma is tightly connected to a circle packing problem, see, e.g., https://en.wikipedia.org/wiki/Circle_packing_in_a_circle. The known results on that problem provide versions of the Lemma for different numbers of disks, which lead to different inequalities of the same kind. E.g., for 4 disks the best possible estimate in the Lemma is $OP_i \geqslant (\sqrt{2} - 1)R_j$, while for 13 disks it has the form $OP_i \geqslant \sqrt{5}\, R_j$. Arguing as in the above solution, one obtains the inequalities

$$\sum_{i=1}^{n} OP_i \geqslant (\sqrt{2} - 1) \sum_{j=4}^{n} R_j \quad \text{and} \quad \sum_{i=1}^{n} OP_i \geqslant \sqrt{5} \sum_{j=13}^{n} R_j.$$

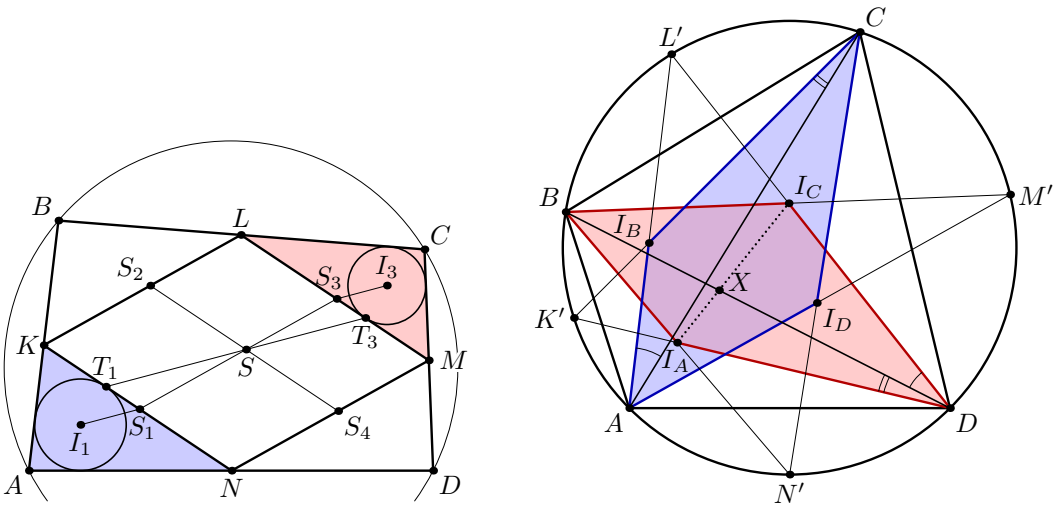
However, there are some harder arguments which allow to improve these inequalities, meaning that the $R_j$ with large indices may be taken with much greater factors.

**G5.** Let $ABCD$ be a cyclic quadrilateral with no two sides parallel. Let $K$, $L$, $M$, and $N$ be points lying on sides $AB$, $BC$, $CD$, and $DA$, respectively, such that $KLMN$ is a rhombus with $KL \parallel AC$ and $LM \parallel BD$. Let $\omega_1$, $\omega_2$, $\omega_3$, and $\omega_4$ be the incircles of triangles $ANK$, $BKL$, $CLM$, and $DMN$, respectively. Prove that the internal common tangents to $\omega_1$ and $\omega_3$ and the internal common tangents to $\omega_2$ and $\omega_4$ are concurrent.

*(Poland)*

**Solution 1.** Let $I_i$ be the center of $\omega_i$, and let $r_i$ be its radius for $i = 1, 2, 3, 4$. Denote by $T_1$ and $T_3$ the points of tangency of $\omega_1$ and $\omega_3$ with $NK$ and $LM$, respectively. Suppose that the internal common tangents to $\omega_1$ and $\omega_3$ meet at point $S$, which is the center of homothety $h$ with negative ratio (namely, with ratio $-\dfrac{r_3}{r_1}$) mapping $\omega_1$ to $\omega_3$. This homothety takes $T_1$ to $T_3$ (since the tangents to $\omega_1$ and $\omega_3$ at $T_1$ to $T_3$ are parallel), hence $S$ is a point on the segment $T_1T_3$ with $T_1S : ST_3 = r_1 : r_3$.

Construct segments $S_1S_3 \parallel KL$ and $S_2S_4 \parallel LM$ through $S$ with $S_1 \in NK$, $S_2 \in KL$, $S_3 \in LM$, and $S_4 \in MN$. Note that $h$ takes $S_1$ to $S_3$, hence $I_1S_1 \parallel I_3S_3$, and $S_1S : SS_3 = r_1 : r_3$. We will prove that $S_2S : SS_4 = r_2 : r_4$ or, equivalently, $KS_1 : S_1N = r_2 : r_4$. This will yield the problem statement; indeed, applying similar arguments to the intersection point $S'$ of the internal common tangents to $\omega_2$ and $\omega_4$, we see that $S'$ satisfies similar relations, and there is a unique point inside $KLMN$ satisfying them. Therefore, $S' = S$.




Further, denote by $I_A$, $I_B$, $I_C$, $I_D$ and $r_A$, $r_B$, $r_C$, $r_D$ the incenters and inradii of triangles $DAB$, $ABC$, $BCD$, and $CDA$, respectively. One can shift triangle $CLM$ by $\overrightarrow{LK}$ to glue it with triangle $AKN$ into a quadrilateral $AKC'N$ similar to $ABCD$. In particular, this shows that $r_1 : r_3 = r_A : r_C$; similarly, $r_2 : r_4 = r_B : r_D$. Moreover, the same shift takes $S_3$ to $S_1$, and it also takes $I_3$ to the incenter $I_3'$ of triangle $KC'N$. Since $I_1S_1 \parallel I_3S_3$, the points $I_1$, $S_1$, $I_3'$ are collinear. Thus, in order to complete the solution, it suffices to apply the following Lemma to quadrilateral $AKC'N$.

*Lemma 1.* Let $ABCD$ be a cyclic quadrilateral, and define $I_A$, $I_C$, $r_B$, and $r_D$ as above. Let $I_AI_C$ meet $BD$ at $X$; then $BX : XD = r_B : r_D$.

*Proof.* Consider an inversion centered at $X$; the images under that inversion will be denoted by primes, e.g., $A'$ is the image of $A$.

By properties of inversion, we have

$$\angle I_C'I_A'D' = \angle XI_A'D' = \angle XDI_A = \angle BDA/2 = \angle BCA/2 = \angle ACI_B.$$

We obtain $\angle I_A'I_C'D' = \angle CAI_B$ likewise; therefore, $\triangle I_C'I_A'D' \sim \triangle ACI_B$. In the same manner, we get $\triangle I_C'I_A'B' \sim \triangle ACI_D$, hence the quadrilaterals $I_C'B'I_A'D'$ and $AI_DCI_B$ are also similar. But the diagonals $AC$ and $I_BI_D$ of quadrilateral $AI_DCI_B$ meet at a point $Y$ such that $I_BY$ :

$YI_D = r_B : r_D$. By similarity, we get $D'X : B'X = r_B : r_D$ and hence $BX : XD = D'X : B'X = r_B : r_D$. □

**Comment 1.** The solution above shows that the problem statement holds also for any *parallelogram* $KLMN$ whose sides are parallel to the diagonals of $ABCD$, as no property specific for a rhombus has been used. This solution works equally well when two sides of quadrilateral $ABCD$ are parallel.

**Comment 2.** The problem may be reduced to Lemma 1 by using different tools, e.g., by using mass point geometry, linear motion of $K$, $L$, $M$, and $N$, etc.

Lemma 1 itself also can be proved in different ways. We present below one alternative proof.

*Proof.* In the circumcircle of $ABCD$, let $K'$, $L'$. $M'$, and $N'$ be the midpoints of arcs $AB$, $BC$, $CD$, and $DA$ containing no other vertices of $ABCD$, respectively. Thus, $K' = CI_B \cap DI_A$, etc. In the computations below, we denote by $[P]$ the area of a polygon $P$. We use similarities $\triangle I_A BK' \sim \triangle I_A DN'$, $\triangle I_B K'L' \sim \triangle I_B AC$, etc., as well as congruences $\triangle I_B K'L' = \triangle BK'L'$ and $\triangle I_D M'N' = \triangle DM'N'$ (e.g., the first congruence holds because $K'L'$ is a common internal bisector of angles $BK'I_B$ and $BL'I_B$).

We have

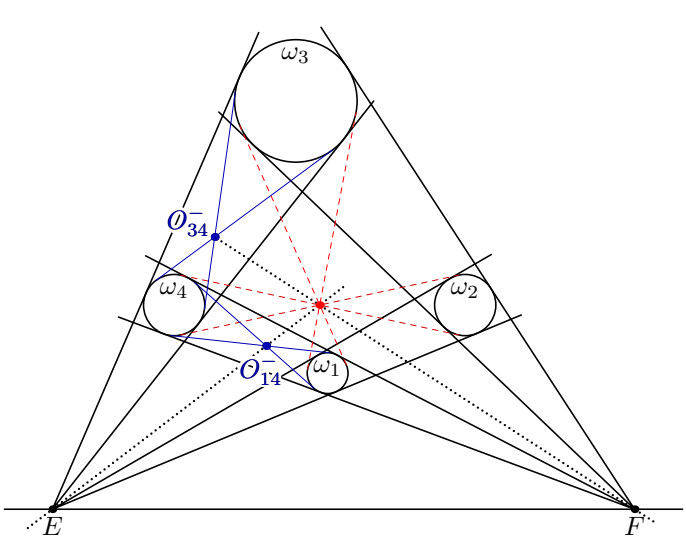
$$\frac{BX}{DX} = \frac{[I_A BI_C]}{[I_A DI_C]} = \frac{BI_A \cdot BI_C \cdot \sin I_A BI_C}{DI_A \cdot DI_C \cdot \sin I_A DI_C} = \frac{BI_A}{DI_A} \cdot \frac{BI_C}{DI_C} \cdot \frac{\sin N'BM'}{\sin K'DL'}$$
$$= \frac{BK'}{DN'} \cdot \frac{BL'}{DM'} \cdot \frac{\sin N'DM'}{\sin K'BL'} = \frac{BK' \cdot BL' \cdot \sin K'BL'}{DN' \cdot DM' \cdot \sin N'DM'} \cdot \frac{\sin^2 N'DM'}{\sin^2 K'BL'}$$
$$= \frac{[K'BL']}{[M'DN']} \cdot \frac{N'M'^2}{K'L'^2} = \frac{[K'I_B L'] \cdot \frac{A'C'^2}{K'L'^2}}{[M'I_D N'] \cdot \frac{A'C'^2}{N'M'^2}} = \frac{[AI_B C]}{[AI_D C]} = \frac{r_B}{r_D},$$

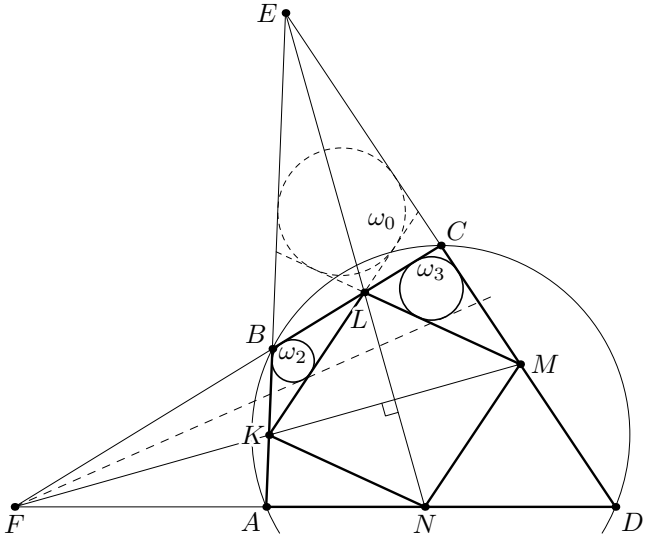
as required. □

**Solution 2.** This solution is based on the following general Lemma.

*Lemma 2.* Let $E$ and $F$ be distinct points, and let $\omega_i$, $i = 1, 2, 3, 4$, be circles lying in the same halfplane with respect to $EF$. For distinct indices $i, j \in \{1, 2, 3, 4\}$, denote by $O_{ij}^+$ (respectively, $O_{ij}^-$) the center of homothety with positive (respectively, negative) ratio taking $\omega_i$ to $\omega_j$. Suppose that $E = O_{12}^+ = O_{34}^+$ and $F = O_{23}^+ = O_{41}^+$. Then $O_{13}^- = O_{24}^-$.

*Proof.* Applying Monge's theorem to triples of circles $\omega_1, \omega_2, \omega_4$ and $\omega_1, \omega_3, \omega_4$, we get that both points $O_{24}^-$ and $O_{13}^-$ lie on line $EO_{14}^-$. Notice that this line is distinct from $EF$. Similarly we obtain that both points $O_{24}^-$ and $O_{13}^-$ lie on $FO_{34}^-$. Since the lines $EO_{14}^-$ and $FO_{34}^-$ are distinct, both points coincide with the meeting point of those lines. □



Turning back to the problem, let $AB$ intersect $CD$ at $E$ and let $BC$ intersect $DA$ at $F$. Assume, without loss of generality, that $B$ lies on segments $AE$ and $CF$. We will show that the points $E$ and $F$, and the circles $\omega_i$ satisfy the conditions of Lemma 2, so the problem statement follows. In the sequel, we use the notation of $O_{ij}^{\pm}$ from the statement of Lemma 2, applied to circles $\omega_1$, $\omega_2$, $\omega_3$, and $\omega_4$.

Using the relations $\triangle ECA \sim \triangle EBD$, $KN \parallel BD$, and $MN \parallel AC$. we get

$$\frac{AN}{ND} = \frac{AN}{AD} \cdot \frac{AD}{ND} = \frac{KN}{BD} \cdot \frac{AC}{NM} = \frac{AC}{BD} = \frac{AE}{ED}.$$

Therefore, by the angle bisector theorem, point $N$ lies on the internal angle bisector of $\angle AED$. We prove similarly that $L$ also lies on that bisector, and that the points $K$ and $M$ lie on the internal angle bisector of $\angle AFB$.

Since $KLMN$ is a rhombus, points $K$ and $M$ are symmetric in line $ELN$. Hence, the convex quadrilateral determined by the lines $EK$, $EM$, $KL$, and $ML$ is a kite, and therefore it has an incircle $\omega_0$. Applying Monge's theorem to $\omega_0$, $\omega_2$, and $\omega_3$, we get that $O_{23}^{+}$ lies on $KM$. On the other hand, $O_{23}^{+}$ lies on $BC$, as $BC$ is an external common tangent to $\omega_2$ and $\omega_3$. It follows that $F = O_{23}^{+}$. Similarly, $E = O_{12}^{+} = O_{34}^{+}$, and $F = O_{41}^{+}$.

**Comment 3.** The reduction to Lemma 2 and the proof of Lemma 2 can be performed with the use of different tools, e.g., by means of Menelaus theorem, by projecting harmonic quadruples, by applying Monge's theorem in some other ways, etc.

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**G6.** Let $I$ and $I_A$ be the incenter and the $A$-excenter of an acute-angled triangle $ABC$ with $AB < AC$. Let the incircle meet $BC$ at $D$. The line $AD$ meets $BI_A$ and $CI_A$ at $E$ and $F$, respectively. Prove that the circumcircles of triangles $AID$ and $I_AEF$ are tangent to each other.

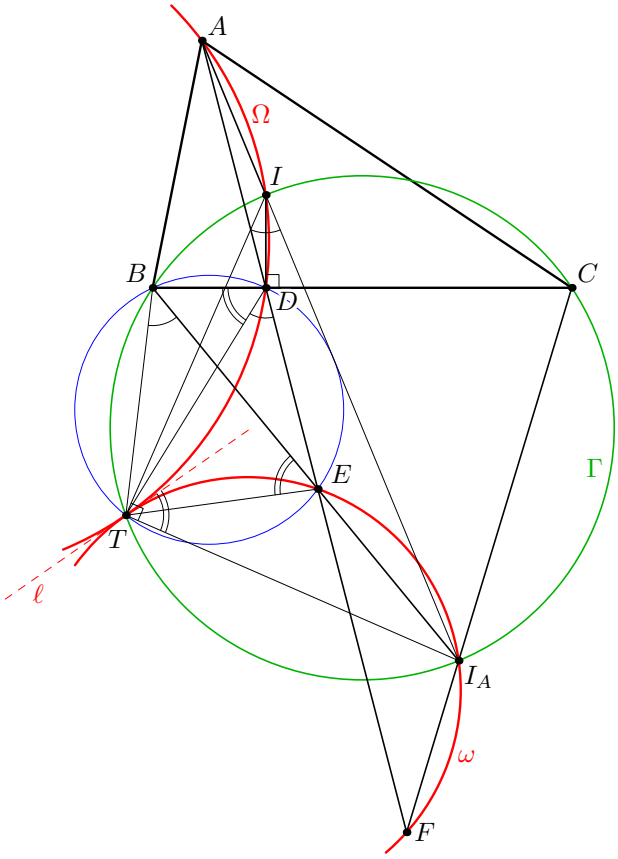
*(Slovakia)*

**Solution 1.** Let $\measuredangle(p, q)$ denote the directed angle between lines $p$ and $q$.

The points $B$, $C$, $I$, and $I_A$ lie on the circle $\Gamma$ with diameter $II_A$. Let $\omega$ and $\Omega$ denote the circles $(I_AEF)$ and $(AID)$, respectively. Let $T$ be the second intersection point of $\omega$ and $\Gamma$. Then $T$ is the Miquel point of the complete quadrilateral formed by the lines $BC$, $BI_A$, $CI_A$, and $DEF$, so $T$ also lies on circle $(BDE)$ (as well as on circle $(CDF)$). We claim that $T$ is a desired tangency point of $\omega$ and $\Omega$.

In order to show that $T$ lies on $\Omega$, use cyclic quadrilaterals $BDET$ and $BII_AT$ to write

$$\measuredangle(DT, DA) = \measuredangle(DT, DE) = \measuredangle(BT, BE) = \measuredangle(BT, BI_A) = \measuredangle(IT, II_A) = \measuredangle(IT, IA).$$




To show that $\omega$ and $\Omega$ are tangent at $T$, let $\ell$ be the tangent to $\omega$ at $T$, so that $\measuredangle(TI_A, \ell) = \measuredangle(EI_A, ET)$. Using circles $(BDET)$ and $(BICI_A)$, we get

$$\measuredangle(EI_A, ET) = \measuredangle(EB, ET) = \measuredangle(DB, DT).$$

Therefore,

$$\measuredangle(TI, \ell) = 90^\circ + \measuredangle(TI_A, \ell) = 90^\circ + \measuredangle(DB, DT) = \measuredangle(DI, DT),$$

which shows that $\ell$ is tangent to $\Omega$ at $T$.

**Solution 2.** We use the notation of circles $\Gamma$, $\omega$, and $\Omega$ as in the previous solution.

Let $L$ be the point opposite to $I$ in circle $\Omega$. Then $\angle IAL = \angle IDL = 90^\circ$, which means that $L$ is the foot of the external bisector of $\angle A$ in triangle $ABC$. Let $LI$ cross $\Gamma$ again at $M$.

Let $T$ be the foot of the perpendicular from $I$ onto $I_AL$. Then $T$ is the second intersection point of $\Gamma$ and $\Gamma$. We will show that $T$ is the desired tangency point.

First, we show that $T$ lies on circle $\omega$. Notice that

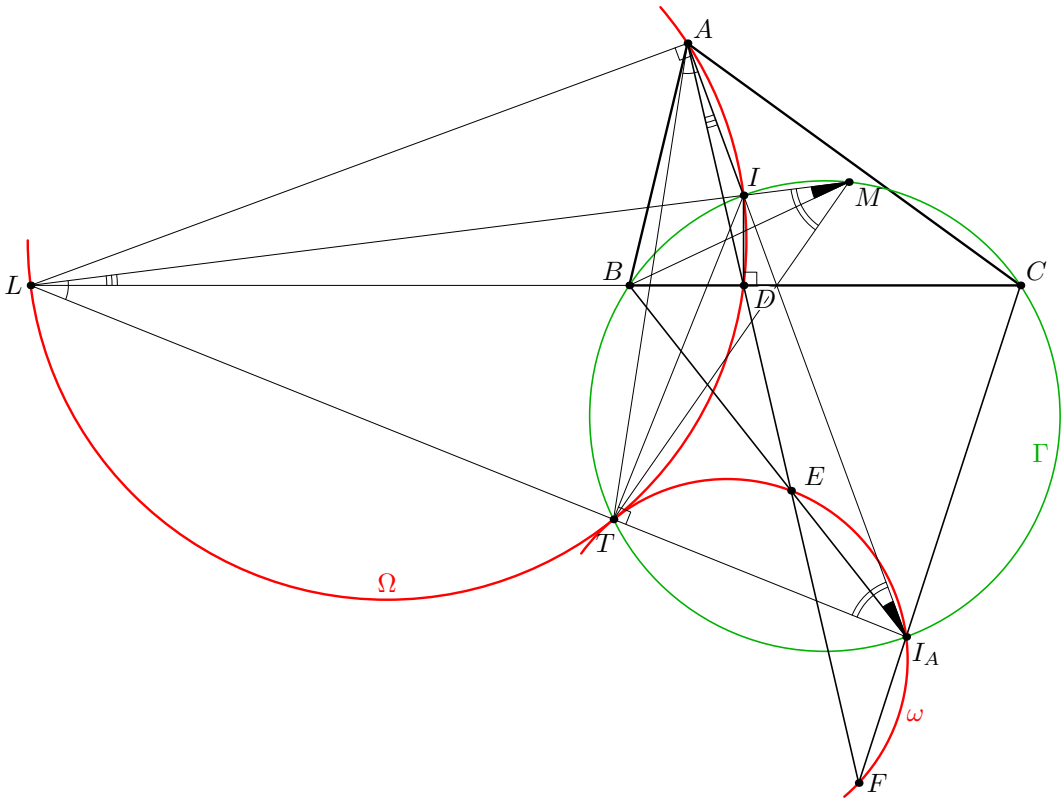
$$\measuredangle(LT, LM) = \measuredangle(AT, AI) \quad \text{and} \quad \measuredangle(MT, ML) = \measuredangle(MT, MI) = \measuredangle(I_AT, I_AI),$$

which shows that triangles $TML$ and $TI_AA$ are similar and equioriented. So there exists a rotational homothety $\tau$ mapping $TML$ to $TI_AA$.

Since $\measuredangle(ML, LD) = \measuredangle(AI, AD)$, we get $\tau(BC) = AD$. Next, since

$$\measuredangle(MB, ML) = \measuredangle(MB, MI) = \measuredangle(I_AB, I_AI) = \measuredangle(I_AE, I_AA),$$

we get $\tau(B) = E$. Similarly, $\tau(C) = F$. Since the points $M$, $B$, $C$, and $T$ are concyclic, so are their $\tau$-images, which means that $T$ lies on $\omega = \tau(\Gamma)$.




Finally, since $\tau(L) = A$ and $\tau(B) = E$, triangles $ATL$ and $ETB$ are similar so that

$$\measuredangle(AT, AL) = \measuredangle(ET, EB) = \measuredangle(EI_A, ET).$$

This means that the tangents to $\Omega$ and $\omega$ at $T$ make the same angle with the line $I_ATL$, so the circles are indeed tangent at $T$.

**Comment.** In both solutions above, a crucial step is a guess that the desired tangency point lies on $\Gamma$. There are several ways to recognize this helpful property.

E.g. one may perform some angle chasing to see that the tangents to $\Omega$ at $L$ and to $\omega$ at $I_A$ are parallel (and the circles lie on different sides of the tangents). This yields that, under the assumption that the circles are tangent externally, the tangency point must lie on $I_AL$. Since $IL$ is a diameter in $\Omega$, this, in turn, implies that $T$ is the projection of $I$ onto $I_AL$.

Another way to see the same fact is to perform a homothety centered at $A$ and mapping $I$ to $I_A$ (and $D$ to some point $D'$). The image $\Omega'$ of $\Omega$ is tangent to $\omega$ at $I_A$, because $\angle BI_AA + \angle CI_AD' = 180^\circ$. This yields that the tangents to $\Omega$ at $I$ and to $\omega$ at $I_A$ are parallel.

There are other ways to describe the tangency point. The next solution presents one of them.

**Solution 3.** We also use the notation of circles $\omega$, and $\Omega$ from the previous solutions.

Perform an inversion centered at $D$. The images of the points will be denoted by primes, e.g., $A'$ is the image of $A$.

For convenience, we use the notation $\angle BID = \beta$, $\angle CID = \gamma$, and $\alpha = 180^\circ - \beta - \gamma = 90^\circ - \angle BAI$. We start with computing angles appearing after inversion. We get

$$\angle DB'I' = \beta, \quad \angle DC'I' = \gamma, \quad \text{and hence} \quad \angle B'I'C' = \alpha;$$
$$\angle E'I'_AF' = \angle E'I'_AD - \angle F'I'_AD = \angle I_AED - \angle I_AFD = \angle EI_AF = 180^\circ - \alpha.$$

Next, we have

$$\angle A'E'B' = \angle DE'B' = \angle DBE = \beta = 90^\circ - \frac{\angle DBA}{2} = 90^\circ - \frac{\angle E'A'B'}{2},$$

which yields that triangle $A'B'E'$ is isosceles with $A'B' = A'E'$. Similarly, $A'F' = A'C'$.

Finally, we get

$$\begin{aligned}\angle A'B'I' = \angle I'B'D - \angle A'B'D = \beta - \angle BAD = \beta - (90^\circ - \alpha) + \angle IAD \\ = \angle ICD + \angle IAD = \angle C'I'D + \angle A'I'D = \angle C'I'A';\end{aligned}$$

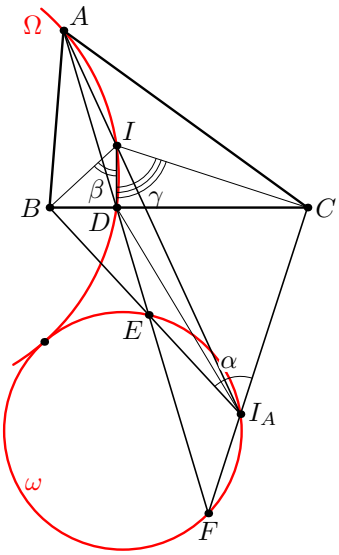
similarly, $\angle A'C'I' = \angle A'I'B'$, so that triangles $A'B'I'$ and $A'I'C'$ are similar. Therefore, $A'I'^2 = A'B' \cdot A'C'$.

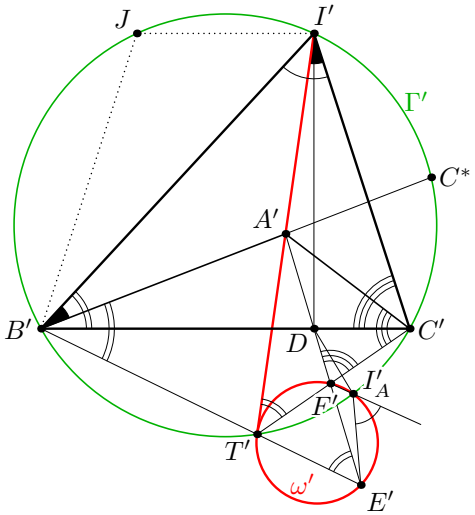
Recall that we need to prove the tangency of line $A'I' = \Omega'$ with circle $(E'F'I'_A) = \omega'$. A desired tangency point $T'$ must satisfy $A'T'^2 = A'E' \cdot A'F'$; the relations obtained above yield

$$A'E' \cdot A'F' = A'B' \cdot A'C' = A'I'^2,$$

so that $T'$ should be symmetric to $I'$ with respect to $A'$.

Thus, let us *define* a point $T'$ as the reflection of $I'$ in $A'$, and show that $T'$ lies on circle $\Omega'$; the equalities above will then imply that $A'T'$ is tangent to $\Omega'$, as desired.




The property that triangles $B'A'I'$ and $I'A'C'$ are similar means that quadrilateral $B'I'C'T'$ is harmonic. Indeed, let $C^*$ be the reflection of $C'$ in the perpendicular bisector of $I'T'$; then $C^*$ lies on $B'A'$ by $\angle B'A'I' = \angle A'I'C' = \angle T'I'C^*$, and then $C^*$ lies on circle $(I'B'T')$ since $A'B' \cdot A'C^* = A'B' \cdot A'C' = A'I'^2 = A'I' \cdot A'T'$. Therefore, $C'$ also lies on that circle (and the circle is $(B'I'C') = \Gamma'$). Moreover, $B'C^*$ is a median in triangle $B'I'T'$, so $B'C'$ is its symmedian, which establishes harmonicity.

Now we have $\angle A'B'T' = \angle I'B'C' = \beta = \angle A'B'E'$; which shows that $E'$ lies on $B'T'$. Similarly, $F'$ lies on $C'T'$. Hence, $\angle E'T'F' = \angle B'I'C' = 180^\circ - \angle E'I'_AF'$, which establishes $T' \in \omega'$.

**Comment 2.** The solution above could be finished without use of harmonicity. E.g., one may notice that both triangles $A'T'F'$ and $A'E'T'$ are similar to triangle $B'I'J$, where $J$ is the point symmetric to $I'$ in the perpendicular bisector of $B'C'$; indeed, we have $\angle T'A'E' = \gamma - \beta = \angle I'B'J'$ and $\frac{B'I'}{B'J'} = \frac{B'I'}{C'I'} = \frac{B'A'}{A'I'} = \frac{A'E'}{A'T'}$. This also allows to compute $\angle E'T'F' = \angle E'T'A' - \angle F'T'A' = \angle I'JB' - \angle JI'B' = \alpha$.

**Comment 3.** Here we list several properties of the configuration in the problem, which can be derived from the solutions above.

The quadrilateral $IBTC$ (as well as $I'B'T'C'$) is harmonic. Hence, line $IT$ contains the meeting point of tangents to $\Gamma$ at $B$ and $C$, i.e., the midpoint $N$ of arc $BAC$ in the circumcircle of $\triangle ABC$.

**G7.** Let $P$ be a point on the circumcircle of an acute-angled triangle $ABC$. Let $D$, $E$, and $F$ be the reflections of $P$ in the midlines of triangle $ABC$ parallel to $BC$, $CA$, and $AB$, respectively. Denote by $\omega_A$, $\omega_B$, and $\omega_C$ the circumcircles of triangles $ADP$, $BEP$, and $CFP$, respectively. Denote by $\omega$ the circumcircle of the triangle formed by the perpendicular bisectors of segments $AD$, $BE$ and $CF$.

Show that $\omega_A$, $\omega_B$, $\omega_C$, and $\omega$ have a common point.

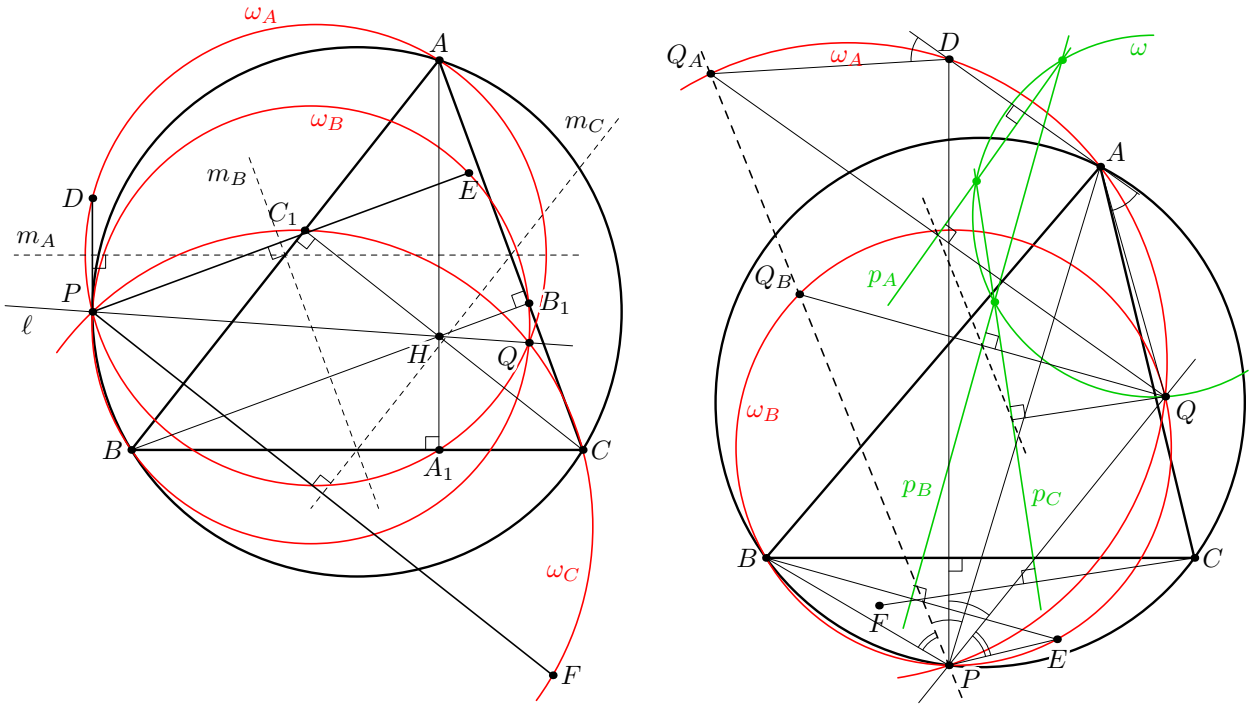
*(Denmark)*

**Solution.** Let $AA_1$, $BB_1$, and $CC_1$ be the altitudes in triangle $ABC$, and let $m_A$, $m_B$, and $m_C$ be the midlines parallel to $BC$, $CA$, and $AB$, respectively. We always denote by $\measuredangle(p, q)$ the directed angle from a line $p$ to a line $q$, taken modulo $180^\circ$.

*Step 1: Circles $\omega_A$, $\omega_B$, and $\omega_C$ share a common point $Q$ different from $P$.*

Notice that $m_A$ is the perpendicular bisector of $PD$, so $\omega_A$ is symmetric with respect to $m_A$. Since $A$ and $A_1$ are also symmetric to each other in $m_A$, the point $A_1$ lies on $\omega_A$. Similarly, $B_1$ and $C_1$ lie on $\omega_B$ and $\omega_C$, respectively.

Let $H$ be the orthocenter of $\triangle ABC$. Quadrilaterals $ABA_1B_1$ and $BCB_1C_1$ are cyclic, so $AH \cdot HA_1 = BH \cdot HB_1 = CH \cdot HC_1$. This means that $H$ lies on pairwise radical axes of $\omega_A$, $\omega_B$, and $\omega_C$. Point $P$ also lies on those radical axes; hence the three circles have a common radical axis $\ell = PH$, and the second meeting point $Q$ of $\ell$ with $\omega_A$ is the second common point of the three circles. Notice here that $H$ lies inside all three circles, hence $Q \neq P$.




*Step 2: Point $Q$ lies on $\omega$.*

Let $p_A$, $p_B$, and $p_C$ denote the perpendicular bisectors of $AD$, $BE$, and $CF$, respectively; denote by $\Delta$ the triangle formed by those perpendicular bisectors. By Simson's theorem, in order to show that $Q$ lies on the circumcircle $\omega$ of $\Delta$, it suffices to prove that the projections of $Q$ onto the sidelines $p_A$, $p_B$, and $p_C$ are collinear. Alternatively, but equivalently, it suffices to prove that the reflections $Q_A$, $Q_B$, and $Q_C$ of $Q$ in those lines, respectively, are collinear. In fact, we will show that four points $P$, $Q_A$, $Q_B$, and $Q_C$ are collinear.

Since $p_A$ is the common perpendicular bisector of $AD$ and $QQ_A$, the point $Q_A$ lies on $\omega_A$, and, moreover, $\measuredangle(DA, DQ_A) = \measuredangle(AQ, AD)$. Therefore,

$$\measuredangle(PA, PQ_A) = \measuredangle(DA, DQ_A) = \measuredangle(AQ, AD) = \measuredangle(PQ, PD) = \measuredangle(PQ, BC) + 90^\circ.$$

Similarly, we get $\measuredangle(PB, PQ_B) = \measuredangle(PQ, CA) + 90^\circ$. Therefore,

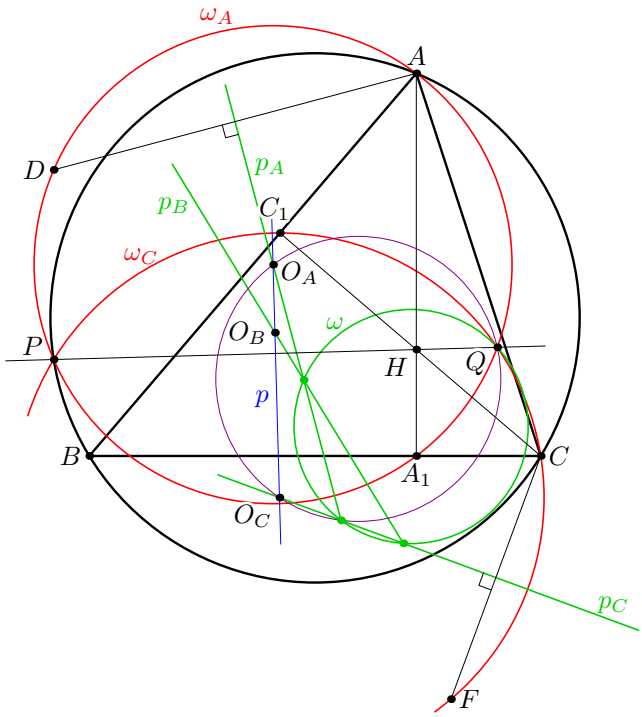
$$\begin{aligned}\measuredangle(PQ_A, PQ_B) &= \measuredangle(PQ_A, PA) + \measuredangle(PA, PB) + \measuredangle(PB, PQ_B)\\ &= \measuredangle(BC, PQ) + 90^\circ + \measuredangle(CA, CB) + \measuredangle(PQ, CA) + 90^\circ = 0,\end{aligned}$$

which shows that $P$, $Q_A$, and $Q_B$ are collinear. Similarly, $Q_C$ also lies on $PQ_A$.

**Comment 1.** There are several variations of Step 2. In particular, let $O_A$, $O_B$, and $O_C$ denote the centers of $\omega_A$, $\omega_B$, and $\omega_C$, respectively; they lie on $p_A$, $p_B$, and $p_C$, respectively. Moreover, all those centers lie on the perpendicular bisector $p$ of $PQ$. Now one can show that $\measuredangle(QO_A, p_A) = \measuredangle(QO_B, p_B) = \measuredangle(QO_C, p_C)$, and then finish by applying generalized Simson's theorem, Alternatively, but equivalently, those relations show that $Q$ is the Miquel point of the lines $p_A$, $p_B$, $p_C$, and $p$.

To establish $\measuredangle(QO_A, p_A) = \measuredangle(QO_C, p_C)$, notice that it is equivalent to $\measuredangle(QO_A, QO_C) = \measuredangle(p_A, p_C)$ which may be obtained, e.g., as follows:

$$\begin{aligned}\measuredangle(QO_A, QO_C) &= \measuredangle(QO_A, p) + \measuredangle(p, QO_C) = \measuredangle(AQ, AP) + \measuredangle(CP, CQ)\\ &= \measuredangle(AQ, CQ) + \measuredangle(CP, AP) = \measuredangle(AQ, PQ) + \measuredangle(PQ, CQ) + \measuredangle(CB, AB)\\ &= \measuredangle(AD, AA_1) + \measuredangle(CC_1, CF) + \measuredangle(AA_1, CC_1) = \measuredangle(AD, CF) = \measuredangle(p_A, p_C).\end{aligned}$$




**Comment 2.** The inversion at $H$ with (negative) power $-AH \cdot HA_1$ maps $P$ to $Q$, and the circumcircle of $\triangle ABC$ to its Euler circle. Therefore, $Q$ lies on that Euler circle.

**G8.** Let $\Gamma$ and $I$ be the circumcircle and the incenter of an acute-angled triangle $ABC$. Two circles $\omega_B$ and $\omega_C$ passing through $B$ and $C$, respectively, are tangent at $I$. Let $\omega_B$ meet the shorter arc $AB$ of $\Gamma$ and segment $AB$ again at $P$ and $M$, respectively. Similarly, let $\omega_C$ meet the shorter arc $AC$ of $\Gamma$ and segment $AC$ again at $Q$ and $N$, respectively. The rays $PM$ and $QN$ meet at $X$, and the tangents to $\omega_B$ and $\omega_C$ at $B$ and $C$, respectively, meet at $Y$.

Prove that the points $A$, $X$, and $Y$ are collinear.

*(Netherlands)*

**Solution 1.** Let $AI$, $BI$, and $CI$ meet $\Gamma$ again at $D$, $E$, and $F$, respectively. Let $\ell$ be the common tangent to $\omega_B$ and $\omega_C$ at $I$. We always denote by $\measuredangle(p, q)$ the directed angle from a line $p$ to a line $q$, taken modulo $180^\circ$.

*Step 1: We show that $Y$ lies on $\Gamma$.*

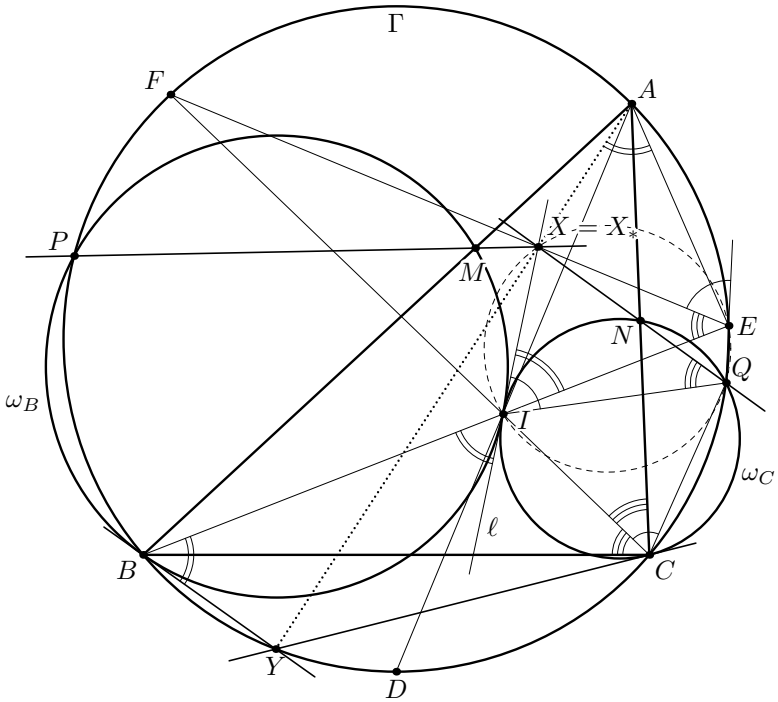
Recall that any chord of a circle makes complementary directed angles with the tangents to the circle at its endpoints. Hence,

$$\measuredangle(BY, BI) + \measuredangle(CI, CY) = \measuredangle(IB, \ell) + \measuredangle(\ell, IC) = \measuredangle(IB, IC).$$

Therefore,

$$\begin{aligned}\measuredangle(BY, BA) + \measuredangle(CA, CY) &= \measuredangle(BI, BA) + \measuredangle(BY, BI) + \measuredangle(CI, CY) + \measuredangle(CA, CI) \\ &= \measuredangle(BC, BI) + \measuredangle(IB, IC) + \measuredangle(CI, CB) = 0,\end{aligned}$$

which yields $Y \in \Gamma$.




*Step 2: We show that $X = \ell \cap EF$.*

Let $X_* = \ell \cap EF$. To prove our claim, it suffices to show that $X_*$ lies on both $PM$ and $QN$; this will yield $X_* = X$. Due to symmetry, it suffices to show $X_* \in QN$.

Notice that

$$\measuredangle(IX_*, IQ) = \measuredangle(CI, CQ) = \measuredangle(CF, CQ) = \measuredangle(EF, EQ) = \measuredangle(EX_*, EQ);$$

therefore, the points $X_*$, $I$, $Q$, and $E$ are concyclic (if $Q = E$, then the direction of $EQ$ is supposed to be the direction of a tangent to $\Gamma$ at $Q$; in this case, the equality means that the circle $(X_*IQ)$ is tangent to $\Gamma$ at $Q$). Then we have

$$\measuredangle(QX_*, QI) = \measuredangle(EX_*, EI) = \measuredangle(EF, EB) = \measuredangle(CA, CF) = \measuredangle(CN, CI) = \measuredangle(QN, QI),$$

which shows that $X_* \in QN$.

*Step 3: We finally show that $A$, $X$, and $Y$ are collinear.*

Recall that $I$ is the orthocenter of triangle $DEF$, and $A$ is symmetric to $I$ with respect to $EF$. Therefore,

$$\measuredangle(AX, AE) = \measuredangle(IE, IX) = \measuredangle(BI, \ell) = \measuredangle(BY, BI) = \measuredangle(BY, BE) = \measuredangle(AY, AE),$$

which yields the desired collinearity.

**Comment 1.** Step 2 in the above solution seems to be crucial. After it has been performed (even without Step 1), there are different ways of finishing the solution.

E.g., one may involve the notion of *isogonal conjugacy*. Let $X_1$ and $Y_1$ be isogonal conjugates of $X$ and $Y$, respectively, with respect to triangle $ABC$. Since $XA = XI$, triangle $AIX$ is isosceles, and hence the lines $AX$ and $XI$ form equal angles with the internal bisector $AI$ of $\angle BAC$. This means that $AX_1 \parallel XI$, or $AX_1 \parallel \ell$.

On the other hand, the lines $BY$ and $\ell$ form equal angles with $BI$, so that $BY_1 \parallel \ell$. Similarly, $CY_1 \parallel \ell$. This means that $Y_1$ is an ideal point, and $AY_1 \parallel \ell$ as well. Therefore, points $A$, $X_1$, and $Y_1$ are collinear, and hence $A$, $X$, and $Y$ are such.

**Solution 2.** Perform an inversion centered at $I$; the images of the points are denoted by primes, e.g., $A'$ is the image of $A$.

On the inverted figure, $I$ and $\Gamma'$ are the orthocenter and the circumcircle of triangle $A'B'C'$, respectively. The points $P'$ and $Q'$ lie on $\Gamma'$ such that $B'P' \parallel C'Q'$ (since $B'P' = \omega'_B$ and $C'Q' = \omega'_C$). The points $M'$ and $N'$ are the second intersections of lines $B'P'$ and $C'Q'$ with the circumcircles $\gamma_B$ and $\gamma_C$ of triangles $A'IB'$ and $A'IC'$, respectively. Notice here that $\gamma_C$ is obtained from $\gamma_B$ by the translation at $\overrightarrow{B'C'}$; the same translation maps line $B'P'$ to $C'Q'$, and hence $M'$ to $N'$. In other words, $B'M'N'C'$ is a parallelogram, and $P'Q'$ partitions it into two isosceles trapezoids.

Point $X'$ is the second intersection point of circles $(IP'M')$ and $(IQ'N')$ that is — the reflection of $I$ in their line of centers. But the centers lie on the common perpendicular bisector $p$ of $P'M'$ and $Q'N'$, so $p$ is that line of centers. Hence, $IX' \parallel B'P'$, as both lines are perpendicular to $p$.

Finally, the point $Y$ satisfies $\measuredangle(BY, BI) = \measuredangle(PB, PI)$ and $\measuredangle(CY, CI) = \measuredangle(QC, QI)$, which yields $\measuredangle(Y'B', Y'I) = \measuredangle(B'P', B'I)$ and $\measuredangle(Y'C', Y'I) = \measuredangle(C'Q', C'I)$. Therefore,

$$\measuredangle(Y'B', Y'C') = \measuredangle(B'P', B'I) + \measuredangle(C'I, C'Q') = \measuredangle(C'I, B'I) = \measuredangle(A'B', A'C'),$$

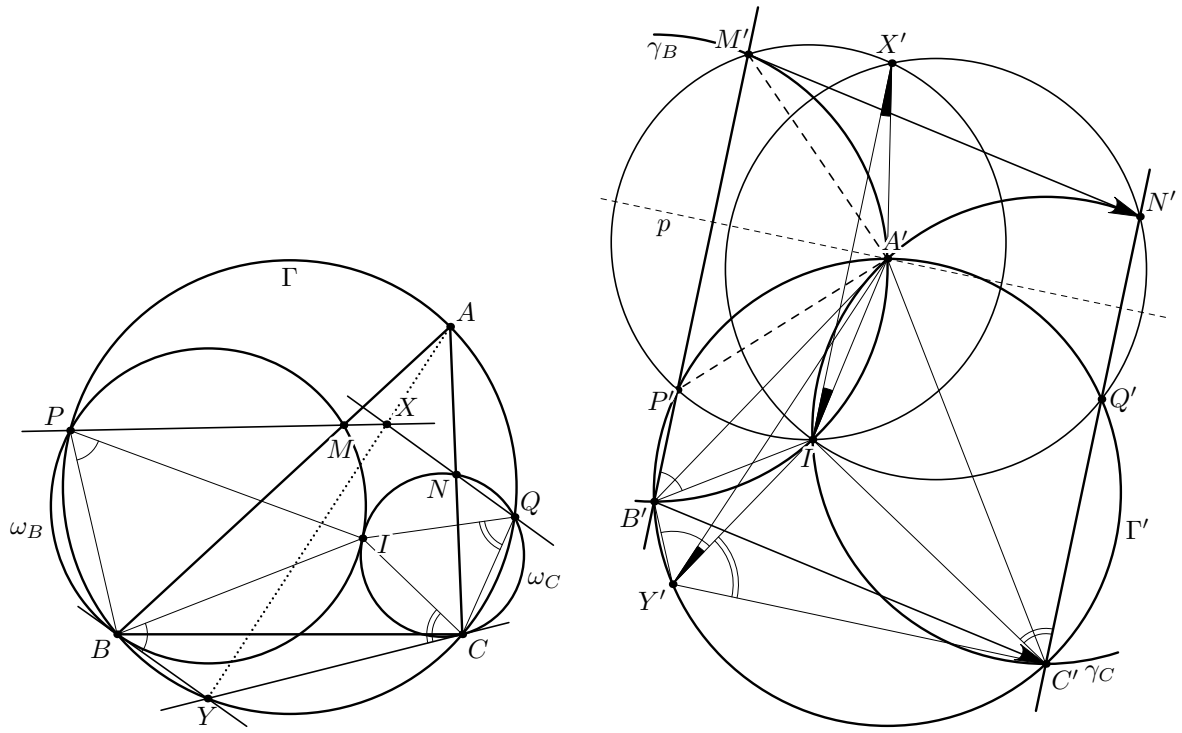
which shows that $Y' \in \Gamma'$.

In congruent circles $\Gamma'$ and $\gamma_B$, the chords $A'P'$ and $A'M'$ subtend the same angle $\angle A'B'P'$; therefore, $A'P' = A'M'$, and hence $A' \in p$. This yields $A'X' = A'I$, and hence $\measuredangle(IA', IX') = \measuredangle(X'I, X'A')$.

Finally, we have

$$\begin{aligned}\measuredangle(Y'I, Y'A') &= \measuredangle(Y'I, Y'B') + \measuredangle(Y'B', Y'A') \\ &= \measuredangle(B'I, B'P') + \measuredangle(IA', IB') = \measuredangle(IA', B'P') = \measuredangle(IA', IX') = \measuredangle(X'I, X'A'),\end{aligned}$$

which yields that the points $A'$, $X'$, $Y'$, and $I$ are concyclic. This means exactly that $A$, $X$, and $Y$ are collinear.




**Comment 2.** An inversion at $I$ may also help in establishing Step 2 in Solution 1. Indeed, relation $A'X' = A'I$ yields $XA = XI$, so that $X \in EF$. On the other hand, $IX' \parallel B'P'$ yields $IX \parallel \ell$, i.e., $X \in \ell$.

**G9.** Prove that there exists a positive constant $c$ such that the following statement is true:

Assume that $n$ is an integer with $n \geqslant 2$, and let $\mathcal{S}$ be a set of $n$ points in the plane such that the distance between any two distinct points in $\mathcal{S}$ is at least 1. Then there is a line $\ell$ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to $\ell$ is at least $cn^{-1/3}$.

(A line $\ell$ *separates* a point set $\mathcal{S}$ if some segment joining two points in $\mathcal{S}$ crosses $\ell$.)

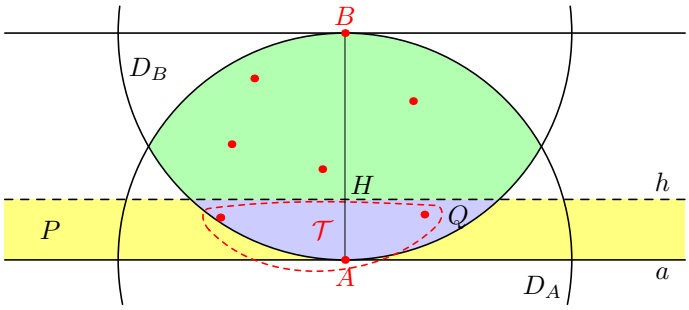
*(Taiwan)*

**Solution.** We prove that the desired statement is true with $c = \frac{1}{8}$. Set $\delta = \frac{1}{8}n^{-1/3}$. For any line $\ell$ and any point $X$, let $X_\ell$ denote the projection of $X$ to $\ell$; a similar notation applies to sets of points.

Suppose that, for some line $\ell$, the set $\mathcal{S}_\ell$ contains two adjacent points $X$ and $Y$ with $XY = 2d$. Then the line perpendicular to $\ell$ and passing through the midpoint of segment $XY$ separates $\mathcal{S}$, and all points in $\mathcal{S}$ are at least $d$ apart from $\ell$. Thus, if $d \geqslant \delta$, then a desired line has been found. For the sake of contradiction, we assume that no such points exist, in any projection.

Choose two points $A$ and $B$ in $\mathcal{S}$ with the maximal distance $M = AB$ (i.e., $AB$ is a *diameter* of $\mathcal{S}$); by the problem condition, $M \geqslant 1$. Denote by $\ell$ the line $AB$. The set $\mathcal{S}$ is contained in the intersection of two disks $D_A$ and $D_B$ of radius $M$ centered at $A$ and $B$, respectively. Hence, the projection $\mathcal{S}_\ell$ is contained in the segment $AB$. Moreover, the points in $\mathcal{S}_\ell$ divide that segment into at most $n-1$ parts, each of length less than $2\delta$. Therefore,

$$M < n \cdot 2\delta. \tag{1}$$




Choose a point $H$ on segment $AB$ with $AH = \frac{1}{2}$. Let $P$ be a strip between the lines $a$ and $h$ perpendicular to $AB$ and passing through $A$ and $H$, respectively; we assume that $P$ contains its boundary, which consists of lines $a$ and $h$. Set $\mathcal{T} = P \cap \mathcal{S}$ and let $t = |\mathcal{T}|$. By our assumption, segment $AH$ contains at least $\left\lceil \frac{1}{2} : (2\delta) \right\rceil$ points of $S_\ell$, which yields

$$t \geqslant \frac{1}{4\delta}. \tag{2}$$

Notice that $\mathcal{T}$ is contained in $Q = P \cap D_B$. The set $Q$ is a circular segment, and its projection $Q_a$ is a line segment of length

$$2\sqrt{M^2 - \left(M - \frac{1}{2}\right)^2} < 2\sqrt{M}.$$

On the other hand, for any two points $X, Y \in \mathcal{T}$, we have $XY \geqslant 1$ and $X_\ell Y_\ell \leqslant \frac{1}{2}$, so $X_a Y_a = \sqrt{XY^2 - X_\ell Y_\ell^2} \geqslant \frac{\sqrt{3}}{2}$. To summarize, $t$ points constituting $\mathcal{T}_a$ lie on the segment of length less than $2\sqrt{M}$, and are at least $\frac{\sqrt{3}}{2}$ apart from each other. This yields $2\sqrt{M} > (t-1)\frac{\sqrt{3}}{2}$, or

$$t < 1 + \frac{4\sqrt{M}}{\sqrt{3}} < 4\sqrt{M}, \tag{3}$$

as $M \geqslant 1$.

Combining the estimates (1), (2), and (3), we finally obtain

$$\frac{1}{4\delta} \leqslant t < 4\sqrt{M} < 4\sqrt{2n\delta}, \quad \text{or} \quad 512n\delta^3 > 1,$$

which does not hold for the chosen value of $\delta$.

**Comment 1.** As the proposer mentions, the exponent $-1/3$ in the problem statement is optimal. In fact, for any $n \geqslant 2$, there is a configuration $\mathcal{S}$ of $n$ points in the plane such that any two points in $\mathcal{S}$ are at least 1 apart, but every line $\ell$ separating $\mathcal{S}$ is at most $c'n^{-1/3}\log n$ apart from some point in $\mathcal{S}$; here $c'$ is some absolute constant.

The original proposal suggested to prove the estimate of the form $cn^{-1/2}$. That version admits much easier solutions. E.g., setting $\delta = \frac{1}{16}n^{-1/2}$ and applying (1), we see that $\mathcal{S}$ is contained in a disk $D$ of radius $\frac{1}{8}n^{1/2}$. On the other hand, for each point $X$ of $\mathcal{S}$, let $D_X$ be the disk of radius $\frac{1}{2}$ centered at $X$; all these disks have disjoint interiors and lie within the disk concentric to $D$, of radius $\frac{1}{16}n^{1/2} + \frac{1}{2} < \frac{1}{2}n^{1/2}$. Comparing the areas, we get

$$n \cdot \frac{\pi}{4} \leqslant \pi\left(\frac{n^{1/2}}{16} + \frac{1}{2}\right)^2 < \frac{\pi n}{4},$$

which is a contradiction.

The Problem Selection Committee decided to choose a harder version for the Shortlist.

**Comment 2.** In this comment, we discuss some versions of the solution above, which avoid concentrating on the diameter of $\mathcal{S}$. We start with introducing some terminology suitable for those versions.

Put $\delta = cn^{-1/3}$ for a certain sufficiently small positive constant $c$. For the sake of contradiction, suppose that, for some set $\mathcal{S}$ satisfying the conditions in the problem statement, there is no separating line which is at least $\delta$ apart from each point of $\mathcal{S}$.

Let $C$ be the convex hull of $\mathcal{S}$. A line is separating if and only if it meets $C$ (we assume that a line passing through a point of $\mathcal{S}$ is always separating). Consider a strip between two parallel separating lines $a$ and $a'$ which are, say, $\frac{1}{4}$ apart from each other. Define a *slice* determined by the strip as the intersection of $\mathcal{S}$ with the strip. The *length* of the slice is the diameter of the projection of the slice to $a$.

In this terminology, the arguments used in the proofs of (2) and (3) show that for any slice $\mathcal{T}$ of length $L$, we have

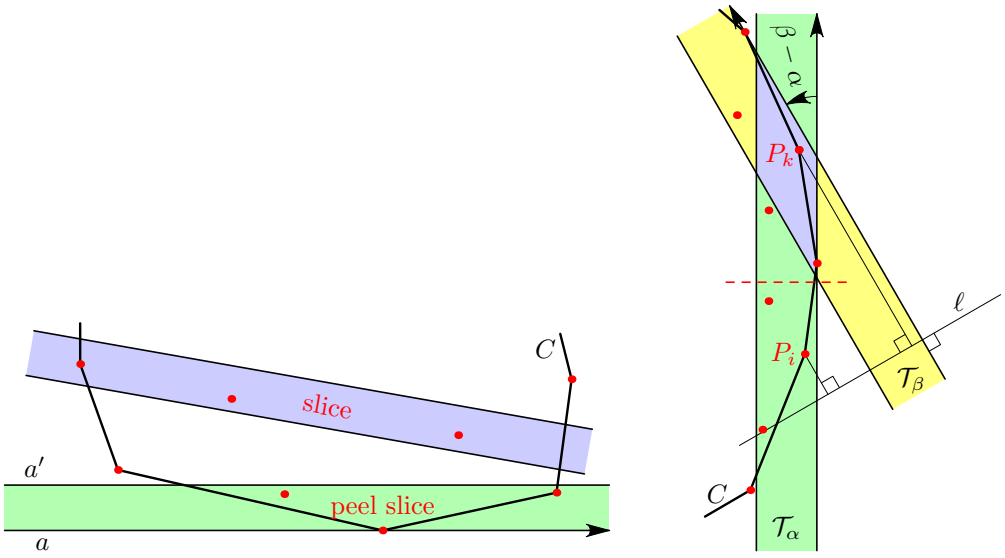
$$\frac{1}{8\delta} \leqslant |\mathcal{T}| \leqslant 1 + \frac{4}{\sqrt{15}}L. \tag{4}$$

The key idea of the solution is to apply these estimates to a *peel slice*, where line $a$ does not cross the interior of $C$. In the above solution, this idea was applied to one carefully chosen peel slice. Here, we outline some different approach involving many of them. We always assume that $n$ is sufficiently large.

Consider a peel slice determined by lines $a$ and $a'$, where $a$ contains no interior points of $C$. We orient $a$ so that $C$ lies to the left of $a$. Line $a$ is called a *supporting line* of the slice, and the obtained direction is the *direction* of the slice; notice that the direction determines uniquely the supporting line and hence the slice. Fix some direction $\mathbf{v}_0$, and for each $\alpha \in [0, 2\pi)$ denote by $\mathcal{T}_\alpha$ the peel slice whose direction is $\mathbf{v}_0$ rotated by $\alpha$ counterclockwise.

When speaking about the slice, we always assume that the figure is rotated so that its direction is vertical from the bottom to the top; then the points in $\mathcal{T}$ get a natural order from the bottom to the top. In particular, we may speak about the *top half* $\mathsf{T}(\mathcal{T})$ consisting of $\lfloor|\mathcal{T}|/2\rfloor$ topmost points in $\mathcal{T}$, and similarly about its *bottom half* $\mathsf{B}(\mathcal{T})$. By (4), each half contains at least 10 points when $n$ is large.

*Claim.* Consider two angles $\alpha, \beta \in [0, \pi/2]$ with $\beta - \alpha \geqslant 40\delta =: \phi$. Then all common points of $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ lie in $\mathsf{T}(\mathcal{T}_\alpha) \cap \mathsf{B}(\mathcal{T}_\beta)$.




*Proof.* By symmetry, it suffices to show that all those points lie in $\mathsf{T}(\mathcal{T}_\alpha)$. Let $a$ be the supporting line of $\mathcal{T}_\alpha$, and let $\ell$ be a line perpendicular to the direction of $\mathcal{T}_\beta$. Let $P_1, \ldots, P_k$ list all points in $\mathcal{T}_\alpha$, numbered from the bottom to the top; by (4), we have $k \geqslant \frac{1}{8}\delta^{-1}$.

Introduce the Cartesian coordinates so that the (oriented) line $a$ is the $y$-axis. Let $P_i$ be any point in $\mathsf{B}(\mathcal{T}_\alpha)$. The difference of ordinates of $P_k$ and $P_i$ is at least $\frac{\sqrt{15}}{4}(k-i) > \frac{1}{3}k$, while their abscissas differ by at most $\frac{1}{4}$. This easily yields that the projections of those points to $\ell$ are at least

$$\frac{k}{3}\sin\phi - \frac{1}{4} \geqslant \frac{1}{24\delta}\cdot 20\delta - \frac{1}{4} > \frac{1}{4}$$

apart from each other, and $P_k$ is closer to the supporting line of $\mathcal{T}_\beta$ than $P_i$, so that $P_i$ does not belong to $\mathcal{T}_\beta$. □

Now, put $\alpha_i = 40\delta i$, for $i = 0, 1, \ldots, \lfloor \frac{1}{40}\delta^{-1}\cdot\frac{\pi}{2}\rfloor$, and consider the slices $\mathcal{T}_{\alpha_i}$. The Claim yields that each point in $\mathcal{S}$ is contained in at most two such slices. Hence, the union $\mathcal{U}$ of those slices contains at least

$$\frac{1}{2}\cdot\frac{1}{8\delta}\cdot\frac{1}{40\delta}\cdot\frac{\pi}{2} = \frac{\lambda}{\delta^2}$$

points (for some constant $\lambda$), and each point in $\mathcal{U}$ is at most $\frac{1}{4}$ apart from the boundary of $C$.

It is not hard now to reach a contradiction with (1). E.g., for each point $X \in \mathcal{U}$, consider a closest point $f(X)$ on the boundary of $C$. Obviously, $f(X)f(Y) \geqslant XY - \frac{1}{2} \geqslant \frac{1}{2}XY$ for all $X, Y \in \mathcal{U}$. This yields that the perimeter of $C$ is at least $\mu\delta^{-2}$, for some constant $\mu$, and hence the diameter of $\mathcal{S}$ is of the same order.

Alternatively, one may show that the projection of $\mathcal{U}$ to the line at the angle of $\pi/4$ with $\mathbf{v}_0$ has diameter at least $\mu\delta^{-2}$ for some constant $\mu$.

# Number Theory

**N1.** Given a positive integer $k$, show that there exists a prime $p$ such that one can choose distinct integers $a_1, a_2, \ldots, a_{k+3} \in \{1, 2, \ldots, p-1\}$ such that $p$ divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, \ldots, k$.

*(South Africa)*

**Solution.** First we choose distinct positive rational numbers $r_1, \ldots, r_{k+3}$ such that

$$r_i r_{i+1} r_{i+2} r_{i+3} = i \quad \text{for } 1 \leqslant i \leqslant k.$$

Let $r_1 = x$, $r_2 = y$, $r_3 = z$ be some distinct primes greater than $k$; the remaining terms satisfy $r_4 = \frac{1}{r_1 r_2 r_3}$ and $r_{i+4} = \frac{i+1}{i} r_i$. It follows that if $r_i$ are represented as irreducible fractions, the numerators are divisible by $x$ for $i \equiv 1 \pmod 4$, by $y$ for $i \equiv 2 \pmod 4$, by $z$ for $i \equiv 3 \pmod 4$ and by none for $i \equiv 0 \pmod 4$. Notice that $r_i < r_{i+4}$; thus the sequences $r_1 < r_5 < r_9 < \ldots$, $r_2 < r_6 < r_{10} < \ldots$, $r_3 < r_7 < r_{11} < \ldots$, $r_4 < r_8 < r_{12} < \ldots$ are increasing and have no common terms, that is, all $r_i$ are distinct.

If each $r_i$ is represented by an irreducible fraction $\frac{u_i}{v_i}$, choose a prime $p$ which divides neither $v_i$, $1 \leqslant i \leqslant k+1$, nor $v_i v_j (r_i - r_j) = v_j u_i - v_i u_j$ for $i < j$, and define $a_i$ by the congruence $a_i v_i \equiv u_i \pmod p$. Since $r_i r_{i+1} r_{i+2} r_{i+3} = i$, we have

$$\begin{aligned} i v_i v_{i+1} v_{i+2} v_{i+3} &= r_i v_i r_{i+1} v_{i+1} r_{i+2} v_{i+2} r_{i+3} v_{i+3} \\ &= u_i u_{i+1} u_{i+2} u_{i+3} \equiv a_i v_i a_{i+1} v_{i+1} a_{i+2} v_{i+2} a_{i+3} v_{i+3} \pmod p \end{aligned}$$

and therefore $a_i a_{i+1} a_{i+2} a_{i+3} \equiv i \pmod p$ for $1 \leqslant i \leqslant k$.

If $a_i \equiv a_j \pmod p$, then $u_i v_j \equiv a_i v_i v_j \equiv u_j v_i \pmod p$, a contradiction.

**Comment.** One can explicitly express residues $b_i \equiv a_1 a_2 \cdot \ldots \cdot a_i \pmod p$ in terms of $b_1$, $b_2$, $b_3$ and $b_0 = 1$:

$$b_{i+3} = i(i-4)(i-8) \cdot \ldots \cdot (i-4k+4) b_r,$$

where $i + 3 = 4k + r$, $0 \leqslant r < 4$. Then the numbers $a_i$ are found from the congruences $b_{i-1} a_i \equiv b_i \pmod p$, and choosing $p$ so that $a_i$ are not congruent modulo $p$ is done in a way very similar to the above solution.

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**N2.** For each prime $p$, there is a kingdom of $p$-Landia consisting of $p$ islands numbered $1, 2, \ldots, p$. Two distinct islands numbered $n$ and $m$ are connected by a bridge if and only if $p$ divides $(n^2 - m + 1)(m^2 - n + 1)$. The bridges may pass over each other, but cannot cross. Prove that for infinitely many $p$ there are two islands in $p$-Landia not connected by a chain of bridges.

*(Denmark)*

**Solution 1.** We prove that for each prime $p > 3$ dividing a number of the form $x^2 - x + 1$ with integer $x$ there are two unconnected islands in $p$-Landia.

For brevity's sake, when a bridge connects the islands numbered $m$ and $n$, we shall speak simply that it connects $m$ and $n$.

A bridge connects $m$ and $n$ if $n \equiv m^2 + 1 \pmod p$ or $m \equiv n^2 + 1 \pmod p$. If $m^2 + 1 \equiv n \pmod p$, we draw an arrow starting at $m$ on the bridge connecting $m$ and $n$. Clearly only one arrow starts at $m$ if $m^2 + 1 \not\equiv m \pmod p$, and no arrows otherwise. The total number of bridges does not exceed the total number of arrows.

Suppose $x^2 - x + 1 \equiv 0 \pmod p$. We may assume that $1 \leqslant x \leqslant p$; then there is no arrow starting at $x$. Since $(1 - x)^2 - (1 - x) + 1 = x^2 - x + 1$, $(p + 1 - x)^2 + 1 \equiv (p + 1 - x) \pmod p$, and there is also no arrow starting at $p + 1 - x$. If $x = p + 1 - x$, that is, $x = \frac{p+1}{2}$, then $4(x^2 - x + 1) = p^2 + 3$ and therefore $x^2 - x + 1$ is not divisible by $p$. Thus the islands $x$ and $p + 1 - x$ are different, and no arrows start at either of them. It follows that the total number of bridges in $p$-Landia does not exceed $p - 2$.

Let $1, 2, \ldots, p$ be the vertices of a graph $G_p$, where an edge connects $m$ and $n$ if and only if there is a bridge between $m$ and $n$. The number of vertices of $G_p$ is $p$ and the number of edges is less than $p - 1$. This means that the graph is not connected, which means that there are two islands not connected by a chain of bridges.

It remains to prove that there are infinitely many primes $p$ dividing $x^2 - x + 1$ for some integer $x$. Let $p_1, p_2, \ldots, p_k$ be any finite set of such primes. The number $(p_1 p_2 \cdot \ldots \cdot p_k)^2 - p_1 p_2 \cdot \ldots \cdot p_k + 1$ is greater than 1 and not divisible by any $p_i$; therefore it has another prime divisor with the required property.

**Solution 2.** One can show, by using only arithmetical methods, that for infinitely many $p$, the kingdom of $p$-Ladia contains two islands connected to no other island, except for each other.

Let arrows between islands have the same meaning as in the previous solution. Suppose that positive $a < p$ satisfies the congruence $x^2 - x + 1 \equiv 0 \pmod p$. We have seen in the first solution that $b = p + 1 - a$ satisfies it too, and $b \neq a$ when $p > 3$. It follows that $ab \equiv a(1 - a) \equiv 1 \pmod p$. If an arrow goes from $t$ to $a$, then $t$ must satisfy the congruence $t^2 + 1 \equiv a \equiv a^2 + 1 \pmod p$; the only such $t \neq a$ is $p - a$. Similarly, the only arrow going to $b$ goes from $p - b$. If one of the numbers $p - a$ and $p - b$, say, $p - a$, is not at the end of any arrow, the pair $a$, $p - a$ is not connected with the rest of the islands. This is true if at least one of the congruences $x^2 + 1 \equiv -a$, $x^2 + 1 \equiv -b$ has no solutions, that is, either $-a - 1$ or $-b - 1$ is a quadratic non-residue modulo $p$.

Note that $x^2 - x + 1 \equiv x^2 - (a + b)x + ab \equiv (x - a)(x - b) \pmod p$. Substituting $x = -1$ we get $(-1 - a)(-1 - b) \equiv 3 \pmod p$. If 3 is a quadratic non-residue modulo $p$, so is one of the numbers $-1 - a$ and $-1 - b$.

Thus it is enough to find infinitely many primes $p > 3$ dividing $x^2 - x + 1$ for some integer $x$ and such that 3 is a quadratic non-residue modulo $p$.

If $x^2 - x + 1 \equiv 0 \pmod p$ then $(2x - 1)^2 \equiv -3 \pmod p$, that is, $-3$ is a quadratic residue modulo $p$, so 3 is a quadratic non-residue if and only if $-1$ is also a non-residue, in other words, $p \equiv -1 \pmod 4$.

Similarly to the first solution, let $p_1, \ldots, p_k$ be primes congruent to $-1$ modulo 4 and dividing numbers of the form $x^2 - x + 1$. The number $(2p_1 \cdot \ldots \cdot p_k)^2 - 2p_1 \cdot \ldots \cdot p_k + 1$ is

not divisible by any $p_i$ and is congruent to $-1$ modulo 4, therefore, it has some prime divisor $p \equiv -1 \pmod 4$ which has the required properties.

**N3.** Let $n$ be an integer with $n \geqslant 2$. Does there exist a sequence $(a_1, \ldots, a_n)$ of positive integers with not all terms being equal such that the arithmetic mean of every two terms is equal to the geometric mean of some (one or more) terms in this sequence?

*(Estonia)*

**Answer:** No such sequence exists.

**Solution 1.** Suppose that $a_1, \ldots, a_n$ satisfy the required properties. Let $d = \gcd(a_1 \ldots, a_n)$. If $d > 1$ then replace the numbers $a_1, \ldots, a_n$ by $\frac{a_1}{d}, \ldots, \frac{a_n}{d}$; all arithmetic and all geometric means will be divided by $d$, so we obtain another sequence satisfying the condition. Hence, without loss of generality, we can assume that $\gcd(a_1 \ldots, a_n) = 1$.

We show two numbers, $a_m$ and $a_k$ such that their arithmetic mean, $\frac{a_m+a_k}{2}$ is different from the geometric mean of any (nonempty) subsequence of $a_1 \ldots, a_n$. That proves that there cannot exist such a sequence.

Choose the index $m \in \{1, \ldots, n\}$ such that $a_m = \max(a_1, \ldots, a_n)$. Note that $a_m \geqslant 2$, because $a_1, \ldots, a_n$ are not all equal. Let $p$ be a prime divisor of $a_m$.

Let $k \in \{1, \ldots, n\}$ be an index such that $a_k = \max\{a_i : p \nmid a_i\}$. Due to $\gcd(a_1 \ldots, a_n) = 1$, not all $a_i$ are divisible by $p$, so such a $k$ exists. Note that $a_m > a_k$ because $a_m \geqslant a_k$, $p \mid a_m$ and $p \nmid a_k$.

Let $b = \frac{a_m+a_k}{2}$; we will show that $b$ cannot be the geometric mean of any subsequence of $a_1, \ldots, a_n$.

Consider the geometric mean, $g = \sqrt[t]{a_{i_1} \cdot \ldots \cdot a_{i_t}}$ of an arbitrary subsequence of $a_1, \ldots, a_n$.

If none of $a_{i_1}, \ldots, a_{i_t}$ is divisible by $p$, then they are not greater than $a_k$, so

$$g = \sqrt[t]{a_{i_1} \cdot \ldots \cdot a_{i_t}} \leqslant a_k < \frac{a_m + a_k}{2} = b,$$

and therefore $g \neq b$.

Otherwise, if at least one of $a_{i_1}, \ldots, a_{i_t}$ is divisible by $p$, then $2g = 2\sqrt[t]{a_{i_1} \cdot \ldots \cdot a_{i_t}}$ is either not an integer or is divisible by $p$, while $2b = a_m + a_k$ is an integer not divisible by $p$, so $g \neq b$ again.

**Solution 2.** Like in the previous solution, we assume that the numbers $a_1, \ldots, a_n$ have no common divisor greater than 1. The arithmetic mean of any two numbers in the sequence is half of an integer; on the other hand, it is a (some integer order) root of an integer. This means each pair's mean is an integer, so all terms in the sequence must be of the same parity; hence they all are odd. Let $d = \min\left\{\gcd(a_i, a_j) \colon a_i \neq a_j\right\}$. By reordering the sequence we can assume that $\gcd(a_1, a_2) = d$, the sum $a_1 + a_2$ is maximal among such pairs, and $a_1 > a_2$.

We will show that $\frac{a_1+a_2}{2}$ cannot be the geometric mean of any subsequence of $a_1 \ldots, a_n$.

Let $a_1 = xd$ and $a_2 = yd$ where $x, y$ are coprime, and suppose that there exist some $b_1, \ldots, b_t \in \{a_1, \ldots, a_n\}$ whose geometric mean is $\frac{a_1+a_2}{2}$. Let $d_i = \gcd(a_1, b_i)$ for $i = 1, 2, \ldots, t$ and let $D = d_1 d_2 \cdot \ldots \cdot d_t$. Then

$$D = d_1 d_2 \cdot \ldots \cdot d_t \mid b_1 b_2 \cdot \ldots \cdot b_t = \left(\frac{a_1 + a_2}{2}\right)^t = \left(\frac{x + y}{2}\right)^t d^t.$$

We claim that $D \mid d^t$. Consider an arbitrary prime divisor $p$ of $D$. Let $\nu_p(x)$ denote the exponent of $p$ in the prime factorization of $x$. If $p \mid \frac{x+y}{2}$, then $p \nmid x, y$, so $p$ is coprime with $x$; hence, $\nu_p(d_i) \leqslant \nu_p(a_1) = \nu_p(xd) = \nu_p(d)$ for every $1 \leqslant i \leqslant t$, therefore $\nu_p(D) = \sum_i \nu_p(d_i) \leqslant t\nu_p(d) = \nu_p(d^t)$. Otherwise, if $p$ is coprime to $\frac{x+y}{2}$, we have $\nu_p(D) \leqslant \nu_p(d^t)$ trivially. The claim has been proved.

Notice that $d_i = \gcd(b_i, a_1) \geqslant d$ for $1 \leqslant i \leqslant t$: if $b_i \neq a_1$ then this follows from the definition of $d$; otherwise we have $b_i = a_1$, so $d_i = a_1 \geqslant d$. Hence, $D = d_1 \cdot \ldots \cdot d_t \geqslant d^t$, and the claim forces $d_1 = \ldots = d_t = d$.

Finally, by $\frac{a_1+a_2}{2} > a_2$ there must be some $b_k$ which is greater than $a_2$. From $a_1 > a_2 \geqslant d = \gcd(a_1, b_k)$ it follows that $a_1 \neq b_k$. Now the have a pair $a_1, b_k$ such that $\gcd(a_1, b_k) = d$ but $a_1 + b_k > a_1 + a_2$; that contradicts the choice of $a_1$ and $a_2$.

**Comment.** The original problem proposal contained a second question asking if there exists a non-constant sequence $(a_1, \ldots, a_n)$ of positive integers such that the geometric mean of every two terms is equal the arithmetic mean of some terms.

For $n \geqslant 3$ such a sequence is $(4, 1, 1, \ldots, 1)$. The case $n = 2$ can be done by the trivial estimates

$$\min(a_1, a_2) < \sqrt{a_1 a_2} < \frac{a_1 + a_2}{2} < \max(a_1, a_2).$$

The Problem Selection Committee found this variant less interesting and suggests using only the first question.

**N4.** For any odd prime $p$ and any integer $n$, let $d_p(n) \in \{0, 1, \ldots, p-1\}$ denote the remainder when $n$ is divided by $p$. We say that $(a_0, a_1, a_2, \ldots)$ is a *$p$-sequence*, if $a_0$ is a positive integer coprime to $p$, and $a_{n+1} = a_n + d_p(a_n)$ for $n \geqslant 0$.

(a) Do there exist infinitely many primes $p$ for which there exist $p$-sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ such that $a_n > b_n$ for infinitely many $n$, and $b_n > a_n$ for infinitely many $n$?

(b) Do there exist infinitely many primes $p$ for which there exist $p$-sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ such that $a_0 < b_0$, but $a_n > b_n$ for all $n \geqslant 1$?

*(United Kingdom)*

**Answer:** Yes, for both parts.

**Solution.** Fix some odd prime $p$, and let $T$ be the smallest positive integer such that $p \mid 2^T - 1$; in other words, $T$ is the multiplicative order of 2 modulo $p$.

Consider any $p$-sequence $(x_n) = (x_0, x_1, x_2, \ldots)$. Obviously, $x_{n+1} \equiv 2x_n \pmod p$ and therefore $x_n \equiv 2^n x_0 \pmod p$. This yields $x_{n+T} \equiv x_n \pmod p$ and therefore $d(x_{n+T}) = d(x_n)$ for all $n \geqslant 0$. It follows that the sum $d(x_n) + d(x_{n+1}) + \ldots + d(x_{n+T-1})$ does not depend on $n$ and is thus a function of $x_0$ (and $p$) only; we shall denote this sum by $S_p(x_0)$, and extend the function $S_p(\cdot)$ to all (not necessarily positive) integers. Therefore, we have $x_{n+kT} = x_n + kS_p(x_0)$ for all positive integers $n$ and $k$. Clearly, $S_p(x_0) = S_p(2^t x_0)$ for every integer $t \geqslant 0$.

In both parts, we use the notation

$$S_p^+ = S_p(1) = \sum_{i=0}^{T-1} d_p(2^i) \quad \text{and} \quad S_p^- = S_p(-1) = \sum_{i=0}^{T-1} d_p(p - 2^i).$$

(a) Let $q > 3$ be a prime and $p$ a prime divisor of $2^q + 1$ that is greater than 3. We will show that $p$ is suitable for part (a). Notice that $9 \nmid 2^q + 1$, so that such a $p$ exists. Moreover, for any two odd primes $q < r$ we have $\gcd(2^q + 1, 2^r + 1) = 2^{\gcd(q,r)} + 1 = 3$, thus there exist infinitely many such primes $p$.

For the chosen $p$, we have $T = 2q$. Since $2^q \equiv -1 \pmod p$, we have $S_p^+ = S_p^-$. Now consider the $p$-sequences $(a_n)$ and $(b_n)$ with $a_0 = p + 1$ and $b_0 = p - 1$; we claim that these sequences satisfy the required conditions. We have $a_0 > b_0$ and $a_1 = p + 2 < b_1 = 2p - 2$. It follows then that

$$a_{k \cdot 2q} = a_0 + kS_p^+ > b_0 + kS_p^+ = b_{k \cdot 2q} \quad \text{and} \quad a_{k \cdot 2q+1} = a_1 + kS_p^+ < b_1 + kS_p^+ = b_{k \cdot 2q+1}$$

for all $k = 0, 1, \ldots$, as desired.

(b) Let $q$ be an odd prime and $p$ a prime divisor of $2^q - 1$; thus we have $T = q$. We will show that $p$ is suitable for part (b). Notice that the numbers of the form $2^q - 1$ are pairwise coprime (since $\gcd(2^q - 1, 2^r - 1) = 2^{\gcd(q,r)} - 1 = 1$ for any two distinct primes $q$ and $r$), thus there exist infinitely many such primes $p$. Notice that $d_p(x) + d_p(p - x) = p$ for all $x$ with $p \nmid x$, so that the sum $S_p^+ + S_p^- = pq$ is odd, which yields $S_p^+ = S_p(1) \neq S_p(-1) = S_p^-$.

Assume that $(x_n)$ and $(y_n)$ are two $p$-sequences with $S_p(x_0) > S_p(y_0)$ but $x_0 < y_0$. The first condition yields that

$$x_{Mq+r} - y_{Mq+r} = (x_r - y_r) + M\big(S_p(x_0) - S_p(y_0)\big) \geqslant (x_r - y_r) + M$$

for all nonnegative integers $M$ and every $r = 0, 1, \ldots, q-1$. Thus, we have $x_n > y_n$ for every $n \geqslant q + q \cdot \max\{y_r - x_r \colon r = 0, 1, \ldots, q-1\}$. Now, since $x_0 < y_0$, there exists the largest $n_0$ with $x_{n_0} < y_{n_0}$. In this case the $p$-sequences $a_n = x_{n-n_0}$ and $b_n = y_{n-n_0}$ possess the desired property (notice here that $x_n \neq y_n$ for all $n \geqslant 0$, as otherwise we would have $S_p(x_0) = S_p(x_n) = S_p(y_n) = S_p(y_0)$).

It remains to find $p$-sequences $(x_n)$ and $(y_n)$ satisfying the two conditions. Recall that $S_p^+ \neq S_p^-$. Now, if $S_p^+ > S_p^-$, then we can put $x_0 = 1$ and $y_0 = p - 1$. Otherwise, if $S_p^+ < S_p^-$, then we put $x_0 = p - 1$ and $y_0 = p + 1$.

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**N5.** Determine all functions $f$ defined on the set of all positive integers and taking non-negative integer values, satisfying the three conditions:

*(i)* $f(n) \neq 0$ for at least one $n$;

*(ii)* $f(xy) = f(x) + f(y)$ for every positive integers $x$ and $y$;

*(iii)* there are infinitely many positive integers $n$ such that $f(k) = f(n-k)$ for all $k < n$.

*(Croatia)*

**Answer:** The sought functions are those of the form $f(n) = c \cdot \nu_p(n)$, where $p$ is some prime, $c$ is a nonnegative integer, and $\nu_p(n)$ denotes the exponent of $p$ in the prime decomposition of $n$.

**Solution 1.** If a number $n$ is a product of primes, $n = p_1 p_2 \cdot \ldots \cdot p_k$, then

$$f(n) = f(p_1) + \ldots + f(p_k),$$

in particular, $f(1) = 0$ (since $f(1) = f(1) + f(1)$).

It is also clear that $f(n) = 0$ implies $f(p) = 0$ for all primes $p$ dividing $n$.

Let us call positive integer $n$ *good* if $f(k) = f(n-k)$ for $0 < k < n$. If $n$ is good then each its divisor $d$ is also good; indeed, if $n = dm$ then

$$f(k) = f(mk) - f(m) = f(n-mk) - f(m) = f(m(d-k)) - f(m) = f(d-k)$$

for $0 < k < d$. Thus, good numbers are products of good primes.

It follows immediately from (i) that there exists a prime $p$ such that $f(p) \neq 0$; let $p$ be the smallest such prime. Then $f(r) = 0$ for all $r < p$ (since all prime divisors of $r < p$ are less than $p$). Now every good number $n > p$ must be divisible by $p$. Indeed, if $n = pk + r$ is a good number, $k > 0$, $0 < r < p$, then $f(p) \leqslant f(pk) = f(n - pk) = f(r) = 0$, a contradiction. Since any divisor of a good number is also good, this means that if a divisor $r$ of a good number is not divisible by $p$, it is less than $p$. Thus all good numbers have the form $r \cdot p^k$ with $r < p$. The condition (iii) implies that $k$ can be arbitrarily large, consequently all powers of $p$ are good.

If $q \neq p$ is a prime, $p^{q-1} - 1$ is divisible by $q$ and $p^{q-1}$ is good. Then $f(q) \leqslant f(p^{q-1} - 1) = f(1) = 0$, that is, $f(q) = 0$.

Now we see that $f(n) = \nu_p(n) \cdot c$, where $c = f(p)$. The conditions (i) and (ii) for all such functions with $c \neq 0$ are obvious; the condition (iii) holds for all $n = p^m$, since $\nu_p(p^m - k) = \nu_p(k)$ when $0 < k < p^m$.

**Solution 2.** We use the notion of a good number from the previous solution. As above, we also denote by $\nu_p(n)$ the exponent of a prime $p$ in the prime decomposition of $n$.

Say that a positive integer $k$ is *big* if $f(k) > 0$. Let $\mathcal{B}$ be the set of big primes, and let $p_1 < p_2 < \ldots$ list the elements of $\mathcal{B}$ (this set might be either finite or infinite). By the problem conditions, we have

$$f(n) = \sum_i \nu_{p_i}(n) f(p_i); \tag{1}$$

thus, the big numbers are those divisible by at least one big prime.

For a positive integer $k$, define its *essence* $e(k)$ to be the largest product $e$ of (not necessarily different) big primes such that $e \mid k$. In other words,

$$e(n) = \prod_{p_i \in \mathcal{B}} p_i^{\nu_{p_i}(n)}.$$

This yields that $k/e(k)$ is not big, so $f(k) = f(e(k)) + f(k/e(k)) = f(e(k))$.

*Lemma.* Assume that $n$ is a good number. Then $e(k) = e(n-k)$ for all $k < n$.

*Proof.* Arguing indirectly, choose a minimal $k$ for which the claim of the lemma is violated. Clearly, $k$ is big, as otherwise $f(k) = f(n-k) = 0$ and hence $e(k) = e(n-k) = 1$.

There are $t = k/e(k)$ multiples of $e(k)$ in each of the segments $[1, k]$ and $[n-k, n-1]$. On the other hand, there are $t-1$ such multiples on $[1, k-1]$ — and, by minimality of $k$, on $[n-k+1, n-1]$ as well. This yields that $n-k$ is a multiple of $e(k)$. Therefore,

$$f(e(k)) = f(k) = f(n-k) = f(e(k)) + f\left(\frac{n-k}{e(k)}\right),$$

so the last summand vanishes, hence $\frac{n-k}{e(k)}$ has no big prime divisors, that is, $e(n-k) = e(k)$. This contradicts our choice. □

Back to the problem, assume that $|\mathcal{B}| \geqslant 2$. Take any good number $n > p_1 p_2$, and let $p_1^\alpha$ be the largest power of $p_1$ smaller than $n$, so that $n \leqslant p_1^{\alpha+1} < p_1^\alpha p_2$. By the lemma, $e(n - p_1^\alpha) = e(p_1^\alpha) = p_1^\alpha$, which yields $p_1^\alpha \mid n$. Similarly, $p_2 \mid n$, so that $n \geqslant p_1^\alpha p_2$. This contradiction shows that $|\mathcal{B}| \leqslant 1$, which by (1) yields that $f$ is listed in the answer.

**Solution 3.** We have $f(\prod p_i^{\alpha_i}) = \sum \alpha_i f(p_i)$. Note that

$$f(n-1) + f(n-2) + \ldots + \ldots f(n-k) \geqslant f(1) + \ldots + f(k)$$

for all $k = 1, 2, \ldots, n-1$, since the difference LHS−RHS is just $f(\binom{n-1}{k})$. Assume that $f(p) > 0$. If $f(k) = f(n-k)$ for all $k$, it implies that $\binom{n-1}{k}$ is not divisible by $p$ for all $k = 1, 2, \ldots, n-2$. It is well known that it implies $n = a \cdot p^s$, $a < p$. If there are two primes $p, q$ such that $f(p) > 0, f(q) > 0$, there exist only finitely many $n$ which are equal both to $a \cdot p^s$, $a < p$, and $b \cdot q^t, b < q$. So there exists at most one such $p$, and therefore $f(n) = C \cdot \nu_p(n)$ for some constant $C$.

**Solution 4.** We call a function $f : \mathbb{N} \to \mathbb{N}_0$ satisfying $(ii)$ *additive*. We call a pair $(f, n)$, where $f$ is an additive function and $n \in \mathbb{N}$, *good*, if for all $k < n$ it holds $f(k) = f(n-k)$. For an additive function $f$ and a prime number $p$ the number $\frac{f(p)}{\ln p}$ is denoted by $g(f, p)$.

Let $(f, n)$ be a good pair such that $f(p) > 0$ for at least two primes less than $n$. Let $p_0$ be the prime with maximal $g(f, p)$ among all primes $p < n$. Let $a_0$ be the maximal exponent such that $p_0^{a_0} < n$. Then $f(k) < f(p_0^{a_0})$ for all $k < p_0^{a_0}$. Indeed, if $k = p_1^{a_1} \ldots p_m^{a_m} < p_0^{a_0}$, then

$$\begin{aligned} f(k) &= a_1 f(p_1) + \ldots + a_m f(p_m) = g(f, p_1) a_1 \ln p_1 + \ldots + g(f, p_m) a_m \ln a_m \\ &< g(f, p_0) a_0 \ln p_0 = f(p_0^{a_0}). \end{aligned}$$

Let $n = b p_0^{a_0} + r$, where $0 < r < p_0^{a_0}$. Then $f(r) = f(b p_0^{a_0}) \geqslant f(p_0^{a_0})$. This contradiction shows that $p_0^{a_0} | n$. Then $n = p_0^{\nu_{p_0}(n)} n'$, where $n' \leqslant p_0$.

The functions $f_1(m) := f(p_0)\nu_{p_0}(m)$ and $f_2 := f - f_1$ are additive (obviously $f(m) \geqslant f(p_0^{\nu_{p_0}(m)}) = f_1(m)$, since $p_0^{\nu_{p_0}(m)}$ divides $m$). For $k < n$, $\nu_p(k) = \nu_p(n-k)$. Hence the pair $(f_2, n)$ is also good. Note that $f_2(p_0) = 0$.

Choose among all primes $p < n$ the prime $q_0$ with maximal $g(f_2, p)$. As above we can prove that $n = q_0^{\nu_{q_0}(n)} n''$ with $n'' < q_0$. Since $p_0 \neq q_0$, we get a contradiction. Thus $f(n) = f(p) \cdot \nu_p(n)$.

**N6.** For a positive integer $n$, let $d(n)$ be the number of positive divisors of $n$, and let $\varphi(n)$ be the number of positive integers not exceeding $n$ which are coprime to $n$. Does there exist a constant $C$ such that

$$\frac{\varphi(d(n))}{d(\varphi(n))} \leqslant C$$

for all $n \geqslant 1$?

*(Cyprus)*

**Answer:** No, such constant does not exist.

**Solution 1.** Fix $N > 1$, let $p_1, \ldots, p_k$ be all primes between 1 and $N$ and $p_{k+1}, \ldots, p_{k+s}$ be all primes between $N+1$ and $2N$. Since for $j \leqslant k+s$ all prime divisors of $p_j - 1$ do not exceed $N$, we have

$$\prod_{j=1}^{k+s}(p_j - 1) = \prod_{i=1}^{k} p_i^{c_i},$$

with some fixed exponents $c_1, \ldots, c_k$. Choose a huge prime number $q$ and consider a number

$$n = (p_1 \cdot \ldots \cdot p_k)^{q-1} \cdot (p_{k+1} \cdot \ldots \cdot p_{k+s}).$$

Then

$$\varphi(d(n)) = \varphi(q^k \cdot 2^s) = q^{k-1}(q-1)2^{s-1}$$

and

$$d(\varphi(n)) = d\Big((p_1 \cdot \ldots \cdot p_k)^{q-2} \prod_{i=1}^{k+s}(p_i - 1)\Big) = d\Big(\prod_{i=1}^{k} p_i^{q-2+c_i}\Big) = \prod_{i=1}^{k}(q - 1 + c_i),$$

so

$$\frac{\varphi(d(n))}{d(\varphi(n))} = \frac{q^{k-1}(q-1)2^{s-1}}{\prod_{i=1}^{k}(q-1+c_i)} = 2^{s-1} \cdot \frac{q-1}{q} \cdot \prod_{i=1}^{k} \frac{q}{q-1+c_i},$$

which can be made arbitrarily close to $2^{s-1}$ by choosing $q$ large enough. It remains to show that $s$ can be arbitrarily large, i.e. that there can be arbitrarily many primes between $N$ and $2N$.

This follows, for instance, from the well-known fact that $\sum \frac{1}{p} = \infty$, where the sum is taken over the set $\mathbb{P}$ of prime numbers. Indeed, if, for some constant $C$, there were always at most $C$ primes between $2^\ell$ and $2^{\ell+1}$, we would have

$$\sum_{p \in \mathbb{P}} \frac{1}{p} = \sum_{\ell=0}^{\infty} \sum_{\substack{p \in \mathbb{P} \\ p \in [2^\ell, 2^{\ell+1})}} \frac{1}{p} \leqslant \sum_{\ell=0}^{\infty} \frac{C}{2^\ell} < \infty,$$

which is a contradiction.

**Comment 1.** Here we sketch several alternative elementary self-contained ways to perform the last step of the solution above. In particular, they avoid using divergence of $\sum \frac{1}{p}$.

Suppose that for some constant $C$ and for every $k = 1, 2, \ldots$ there exist at most $C$ prime numbers between $2^k$ and $2^{k+1}$. Consider the prime factorization of the factorial $(2^n)! = \prod p^{\alpha_p}$. We have $\alpha_p = \lfloor 2^n/p \rfloor + \lfloor 2^n/p^2 \rfloor + \ldots$. Thus, for $p \in [2^k, 2^{k+1})$, we get $\alpha_p \leqslant 2^n/2^k + 2^n/2^{k+1} + \ldots = 2^{n-k+1}$, therefore $p^{\alpha_p} \leqslant 2^{(k+1)2^{n-k+1}}$. Combining this with the bound $(2m)! \geqslant m(m+1) \cdot \ldots \cdot (2m-1) \geqslant m^m$ for $m = 2^{n-1}$ we get

$$2^{(n-1)\cdot 2^{n-1}} \leqslant (2^n)! \leqslant \prod_{k=1}^{n-1} 2^{C(k+1)2^{n-k+1}},$$

or

$$\sum_{k=1}^{n-1} C(k+1)2^{1-k} \geqslant \frac{n-1}{2}$$

that fails for large $n$ since $C(k+1)2^{1-k} < 1/3$ for all but finitely many $k$.

In fact, a much stronger inequality can be obtained in an elementary way: Note that the formula for $\nu_p(n!)$ implies that if $p^\alpha$ is the largest power of $p$ dividing $\binom{n}{n/2}$, then $p^\alpha \leqslant n$. By looking at prime factorization of $\binom{n}{n/2}$ we instantaneously infer that

$$\pi(n) \geqslant \log_n \binom{n}{n/2} \geqslant \frac{\log\left(2^n/n\right)}{\log n} \geqslant \frac{n}{2\log n}.$$

This, in particular, implies that for infinitely many $n$ there are at least $\frac{n}{3\log n}$ primes between $n$ and $2n$.

**Solution 2.** In this solution we will use the Prime Number Theorem which states that

$$\pi(m) = \frac{m}{\log m} \cdot (1 + o(1)),$$

as $m$ tends to infinity. Here and below $\pi(m)$ denotes the number of primes not exceeding $m$, and log the natural logarithm.

Let $m > 5$ be a large positive integer and let $n := p_1 p_2 \cdot \ldots \cdot p_{\pi(m)}$ be the product of all primes not exceeding $m$. Then $\varphi(d(n)) = \varphi\left(2^{\pi(m)}\right) = 2^{\pi(m)-1}$. Consider the number

$$\varphi(n) = \prod_{k=1}^{\pi(m)} (p_k - 1) = \prod_{s=1}^{\pi(m/2)} q_s^{\alpha_s},$$

where $q_1, \ldots, q_{\pi(m/2)}$ are primes not exceeding $m/2$. Note that every term $p_k - 1$ contributes at most one prime $q_s > \sqrt{m}$ into the product $\prod_s q_s^{\alpha_s}$, so we have

$$\sum_{s\colon q_s > \sqrt{m}} \alpha_s \leqslant \pi(m) \quad \Longrightarrow \quad \sum_{s\colon q_s > \sqrt{m}} (1 + \alpha_s) \leqslant \pi(m) + \pi(m/2).$$

Hence, applying the AM–GM inequality and the inequality $(A/x)^x \leqslant e^{A/e}$, we obtain

$$\prod_{s\colon q_s > \sqrt{m}} (\alpha_s + 1) \leqslant \left(\frac{\pi(m) + \pi(m/2)}{\ell}\right)^{\ell} \leqslant \exp\left(\frac{\pi(m) + \pi(m/2)}{e}\right),$$

where $\ell$ is the number of primes in the interval $(\sqrt{m}, m]$.

We then use a trivial bound $\alpha_i \leqslant \log_2(\varphi(n)) \leqslant \log_2 n < \log_2(m^m) < m^2$ for each $i$ with $q_i < \sqrt{m}$ to obtain

$$\prod_{s=1}^{\pi(\sqrt{m})} (\alpha_s + 1) \leqslant \left(m^2\right)^{\sqrt{m}} = m^{2\sqrt{m}}.$$

Putting this together we obtain

$$d(\varphi(n)) = \prod_{s=1}^{\pi(m/2)} (\alpha_s + 1) \leqslant \exp\left(2\sqrt{m} \cdot \log m + \frac{\pi(m) + \pi(m/2)}{e}\right).$$

The prime number theorem then implies that

$$\limsup_{m\to\infty} \frac{\log\left(d(\varphi(n))\right)}{m/\log m} \leqslant \limsup_{m\to\infty} \frac{2\sqrt{m} \cdot \log m}{m/\log m} + \limsup_{m\to\infty} \frac{\pi(m) + \pi(m/2)}{e \cdot m/\log m} = \frac{3}{2e}.$$

Whereas, again by prime number theorem, we have

$$\liminf_{m\to\infty} \frac{\log\left(\varphi(d(n))\right)}{m/\log m} = \liminf_{m\to\infty} \frac{\log\left(2^{\pi(m)-1}\right)}{m/\log m} = \log 2.$$

Since $\frac{3}{2e} < \frac{3}{5} < \log 2$, this implies that $\varphi(d(n))/d(\varphi(n))$ can be arbitrarily large.

**Comment 2.** The original formulation of the problem was asking whether $d(\varphi(n)) \geqslant \varphi(d(n))$ for all but finitely many values of $n$. The Problem Selection Committee decided that the presented version is better suited for the Shortlist.

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**N7.** Let $\mathcal{S}$ be a set consisting of $n \geqslant 3$ positive integers, none of which is a sum of two other distinct members of $\mathcal{S}$. Prove that the elements of $\mathcal{S}$ may be ordered as $a_1, a_2, \ldots, a_n$ so that $a_i$ does not divide $a_{i-1} + a_{i+1}$ for all $i = 2, 3, \ldots, n-1$.

*(Ukraine)*

**Common remarks.** In all solutions, we call a set $\mathcal{S}$ of positive integers *good* if no its element is a sum of two other distinct members of $\mathcal{S}$. We will use the following simple observation.

**Observation A.** If $a$, $b$, and $c$ are three distinct elements of a good set $\mathcal{S}$ with $b > a, c$, then $b \nmid a + c$. Otherwise, since $b \neq a + c$, we would have $b \leqslant (a + c)/2 < \max\{a, c\}$.

**Solution 1.** We prove the following stronger statement.

*Claim.* Let $\mathcal{S}$ be a good set consisting of $n \geqslant 2$ positive integers. Then the elements of $\mathcal{S}$ may be ordered as $a_1, a_2, \ldots, a_n$ so that $a_i \nmid a_{i-1} + a_{i+1}$ and $a_i \nmid a_{i-1} - a_{i+1}$, for all $i = 2, 3, \ldots, n-1$.

*Proof.* Say that the ordering $a_1, \ldots, a_n$ of $\mathcal{S}$ is *nice* if it satisfies the required property.

We proceed by induction on $n$. The base case $n = 2$ is trivial, as there are no restrictions on the ordering.

To perform the step of induction, suppose that $n \geqslant 3$. Let $a = \max \mathcal{S}$, and set $\mathcal{T} = \mathcal{S} \setminus \{a\}$. Use the inductive hypothesis to find a nice ordering $b_1, \ldots, b_{n-1}$ of $\mathcal{T}$. We will show that $a$ may be inserted into this sequence so as to reach a nice ordering of $\mathcal{S}$. In other words, we will show that there exists a $j \in \{1, 2, \ldots, n\}$ such that the ordering

$$N_j = (b_1, \ldots, b_{j-1}, a, b_j, b_{j+1}, \ldots, b_{n-1})$$

is nice.

Assume that, for some $j$, the ordering $N_j$ is not nice, so that some element $x$ in it divides either the sum or the difference of two adjacent ones. This did not happen in the ordering of $\mathcal{T}$, hence $x \in \{b_{j-1}, a, b_j\}$ (if, say, $b_{j-1}$ does not exist, then $x \in \{a, b_j\}$; a similar agreement is applied hereafter). But the case $x = a$ is impossible: $a$ cannot divide $b_{j-1} - b_j$, since $0 < |b_{j-1} - b_j| < a$, while $a \nmid b_{j-1} + b_j$ by Observation A. Therefore $x \in \{b_{j-1}, b_j\}$. In this case, assign the number $x$ to the index $j$.

Suppose now that none of the $N_j$ is nice. Since there are $n$ possible indices $j$, and only $n-1$ elements in $\mathcal{T}$, one of those elements (say, $b_k$) is assigned to two different indices, which then should equal $k$ and $k + 1$. This means that $b_k$ divides the numbers $b_{k-1} + \varepsilon_1 a$ and $a + \varepsilon_2 b_{k+1}$, for some signs $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. But then

$$b_{k-1} \equiv -\varepsilon_1 a \equiv \varepsilon_1 \varepsilon_2 b_{k+1} \pmod{b_k},$$

and therefore $b_k \mid b_{k-1} - \varepsilon_1 \varepsilon_2 b_{k+1}$, which means that the ordering of $\mathcal{T}$ was not nice. This contradiction proves the step of induction. □

**Solution 2.** We again prove a stronger statement.

*Claim.* Let $\mathcal{S}$ be an arbitrary set of $n \geqslant 3$ positive integers. Then its elements can be ordered as $a_1, \ldots, a_n$ so that, if $a_i \mid a_{i-1} + a_{i+1}$, then $a_i = \max \mathcal{S}$.

The claim easily implies what we need to prove, due to Observation A.

To prove the Claim, introduce the function $f$ which assigns to any two elements $a, b \in \mathcal{S}$ with $a < b$ the unique integer $f(a, b) \in \{1, 2, \ldots, a\}$ such that $a \mid b + f(a, b)$. Hence, if $b \mid a + c$ for some $a, b, c \in \mathcal{S}$ with $a < b < c$, then $a = f(b, c)$. Therefore, the Claim is a consequence of the following combinatorial lemma.

*Lemma.* Let $\mathcal{S}$ be a set of $n \geqslant 3$ positive integers, and let $f$ be a function which assigns to any $a, b \in \mathcal{S}$ with $a < b$ some integer from the range $\{1, \ldots, a\}$. Then the elements of $\mathcal{S}$ may be ordered as $a_1, a_2, \ldots, a_n$ so as to satisfy the following two conditions simultaneously:

(*i*) **Unimodality:** There exists a $j \in \{1, 2, \ldots, n\}$ such that $a_1 < a_2 < \ldots < a_j > a_{j+1} > \ldots > a_n$; and

(*ii*) **$f$-avoidance:** If $a < b$ are two elements of $\mathcal{S}$, which are adjacent in the ordering, then $f(a, b)$ is not adjacent to $a$.

*Proof.* We call an ordering of $\mathcal{S}$ satisfying (*i*) and (*ii*) *$f$-nice*. We agree that $f(x, y) = x$ for $x \geqslant y$; this agreement puts no extra restriction.

We proceed by induction; for the base case $n = 3$, it suffices to put the maximal element in $\mathcal{S}$ onto the middle position.

To perform the step of induction, let $p < q$ be the two minimal elements of $\mathcal{S}$, and set $\mathcal{T} = \mathcal{S} \setminus \{p\}$. Define a function $g$ by assigning to any elements $a < b$ of $\mathcal{T}$ the value

$$g(a, b) = \begin{cases} q, & \text{if } f(a, b) = p; \\ f(a, b), & \text{otherwise.} \end{cases} \tag{1}$$

Notice that $g(a, b) \leqslant a$ for all $a, b \in \mathcal{T}$.

Use the inductive hypothesis to get a $g$-nice ordering $b_1, b_2, \ldots, b_{n-1}$ of $\mathcal{T}$. By unimodality, either $b_1$ or $b_{n-1}$ equals $q$; these cases differ only by reverting the order, so we assume $b_1 = q$.

Notice that, according to (1), the number $f(b_2, b_3)$ differs from both $p$ and $q$. On the other hand, the number $f(b_{n-1}, b_{n-2})$ differs from at least one of them — say, from $r$; set $s = p + q - r$, so that $\{r, s\} = \{p, q\}$. Now, order $\mathcal{S}$ as

$$s, b_2, b_3, \ldots, b_{n-1}, r.$$

By the induction hypothesis and the above choice, this ordering is nice. □

**Comment.** In the original proposal, the numbers in the set were assumed to be odd (which implies that none is a sum of two others); moreover, the proposal requested to arrange in a row all numbers but one.

On the other hand, Solution 2 shows that the condition of $\mathcal{S}$ being good may be relaxed to the condition that the maximal element of $\mathcal{S}$ is not a sum of two other elements in $\mathcal{S}$. On the other hand, the set $\{1, 2, 3\}$ shows that the condition cannot be merely omitted.

The Problem Selection Committee considered several versions of the problem and chose the best version in their opinion for the Shortlist.

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