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Shortlisted Problems with Solutions

Contents

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Algebra

A1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality

$$f([x]y) = f(x)[f(y)]. \tag{1}$$

holds for all $x, y \in \mathbb{R}$. Here, by $[x]$ we denote the greatest integer not exceeding x.

(France)

Answer. $f(x) = \text{const} = C$, where $C = 0$ or $1 \le C < 2$.

Solution 1. First, setting $x = 0$ in (1) we get

$$f(0) = f(0)[f(y)] \tag{2}$$

for all $y \in \mathbb{R}$. Now, two cases are possible.

Case 1. Assume that $f(0) \ne 0$. Then from (2) we conclude that $[f(y)] = 1$ for all $y \in \mathbb{R}$. Therefore, equation (1) becomes $f([x]y) = f(x)$, and substituting $y = 0$ we have $f(x) = f(0) = C \ne 0$. Finally, from $[f(y)] = 1 = [C]$ we obtain that $1 \le C < 2$.

Case 2. Now we have $f(0) = 0$. Here we consider two subcases.

Subcase 2a. Suppose that there exists $0 < \alpha < 1$ such that $f(\alpha) \ne 0$. Then setting $x = \alpha$ in (1) we obtain $0 = f(0) = f(\alpha)[f(y)]$ for all $y \in \mathbb{R}$. Hence, $[f(y)] = 0$ for all $y \in \mathbb{R}$. Finally, substituting $x = 1$ in (1) provides $f(y) = 0$ for all $y \in \mathbb{R}$, thus contradicting the condition $f(\alpha) \ne 0$.

Subcase 2b. Conversely, we have $f(\alpha) = 0$ for all $0 \le \alpha < 1$. Consider any real z; there exists an integer N such that $\alpha = \dfrac{z}{N} \in [0, 1)$ (one may set $N = [z] + 1$ if $z \ge 0$ and $N = [z] - 1$ otherwise). Now, from (1) we get $f(z) = f([N]\alpha) = f(N)[f(\alpha)] = 0$ for all $z \in \mathbb{R}$.

Finally, a straightforward check shows that all the obtained functions satisfy (1).

Solution 2. Assume that $[f(y)] = 0$ for some y; then the substitution $x = 1$ provides $f(y) = f(1)[f(y)] = 0$. Hence, if $[f(y)] = 0$ for all y, then $f(y) = 0$ for all y. This function obviously satisfies the problem conditions.

So we are left to consider the case when $[f(a)] \ne 0$ for some a. Then we have

$$f([x]a) = f(x)[f(a)], \qquad \text{or} \quad f(x) = \frac{f([x]a)}{[f(a)]}. \tag{3}$$

This means that $f(x_1) = f(x_2)$ whenever $[x_1] = [x_2]$, hence $f(x) = f([x])$, and we may assume that a is an integer.

Now we have

$$f(a) = f\left(2a \cdot \tfrac{1}{2}\right) = f(2a)\left[f\left(\tfrac{1}{2}\right)\right] = f(2a)[f(0)];$$

this implies $[f(0)] \ne 0$, so we may even assume that $a = 0$. Therefore equation (3) provides

$$f(x) = \frac{f(0)}{[f(0)]} = C \ne 0$$

for each x. Now, condition (1) becomes equivalent to the equation $C = C[C]$ which holds exactly when $[C] = 1$.

A2. Let the real numbers a, b, c, d satisfy the relations $a+b+c+d = 6$ and $a^2+b^2+c^2+d^2 = 12$. Prove that

$$36 \leq 4(a^3 + b^3 + c^3 + d^3) - (a^4 + b^4 + c^4 + d^4) \leq 48.$$

(Ukraine)

Solution 1. Observe that

$$\begin{aligned} 4(a^3 + b^3 + c^3 + d^3) - (a^4 + b^4 + c^4 + d^4) &= -\left((a-1)^4 + (b-1)^4 + (c-1)^4 + (d-1)^4\right) \\ &\quad + 6(a^2 + b^2 + c^2 + d^2) - 4(a + b + c + d) + 4 \\ &= -\left((a-1)^4 + (b-1)^4 + (c-1)^4 + (d-1)^4\right) + 52. \end{aligned}$$

Now, introducing $x = a - 1$, $y = b - 1$, $z = c - 1$, $t = d - 1$, we need to prove the inequalities

$$16 \geq x^4 + y^4 + z^4 + t^4 \geq 4,$$

under the constraint

$$x^2 + y^2 + z^2 + t^2 = (a^2 + b^2 + c^2 + d^2) - 2(a + b + c + d) + 4 = 4 \tag{1}$$

(we will not use the value of $x + y + z + t$ though it can be found).

Now the rightmost inequality in (1) follows from the power mean inequality:

$$x^4 + y^4 + z^4 + t^4 \geq \frac{(x^2 + y^2 + z^2 + t^2)^2}{4} = 4.$$

For the other one, expanding the brackets we note that

$$(x^2 + y^2 + z^2 + t^2)^2 = (x^4 + y^4 + z^4 + t^4) + q,$$

where q is a nonnegative number, so

$$x^4 + y^4 + z^4 + t^4 \leq (x^2 + y^2 + z^2 + t^2)^2 = 16,$$

and we are done.

Comment 1. The estimates are sharp; the lower and upper bounds are attained at $(3, 1, 1, 1)$ and $(0, 2, 2, 2)$, respectively.

Comment 2. After the change of variables, one can finish the solution in several different ways. The latter estimate, for instance, it can be performed by moving the variables – since we need only the second of the two shifted conditions.

Solution 2. First, we claim that $0 \leq a, b, c, d \leq 3$. Actually, we have

$$a + b + c = 6 - d, \quad a^2 + b^2 + c^2 = 12 - d^2,$$

hence the power mean inequality

$$a^2 + b^2 + c^2 \geq \frac{(a + b + c)^2}{3}$$

rewrites as

$$12 - d^2 \geq \frac{(6 - d)^2}{3} \quad \iff \quad 2d(d - 3) \leq 0,$$

which implies the desired inequalities for d; since the conditions are symmetric, we also have the same estimate for the other variables.

Now, to prove the rightmost inequality, we use the obvious inequality $x^2(x-2)^2 \geq 0$ for each real x; this inequality rewrites as $4x^3 - x^4 \leq 4x^2$. It follows that

$$(4a^3 - a^4) + (4b^3 - b^4) + (4c^3 - c^4) + (4d^3 - d^4) \leq 4(a^2 + b^2 + c^2 + d^2) = 48,$$

as desired.

Now we prove the leftmost inequality in an analogous way. For each $x \in [0,3]$, we have $(x+1)(x-1)^2(x-3) \leq 0$ which is equivalent to $4x^3 - x^4 \geq 2x^2 + 4x - 3$. This implies that

$$(4a^3 - a^4) + (4b^3 - b^4) + (4c^3 - c^4) + (4d^3 - d^4) \geq 2(a^2 + b^2 + c^2 + d^2) + 4(a+b+c+d) - 12 = 36,$$

as desired.

Comment. It is easy to guess the extremal points $(0,2,2,2)$ and $(3,1,1,1)$ for this inequality. This provides a method of finding the polynomials used in Solution 2. Namely, these polynomials should have the form $x^4 - 4x^3 + ax^2 + bx + c$; moreover, the former polynomial should have roots at 2 (with an even multiplicity) and 0, while the latter should have roots at 1 (with an even multiplicity) and 3. These conditions determine the polynomials uniquely.

Solution 3. First, expanding $48 = 4(a^2 + b^2 + c^2 + d^2)$ and applying the AM–GM inequality, we have

$$\begin{aligned} a^4 + b^4 + c^4 + d^4 + 48 &= (a^4 + 4a^2) + (b^4 + 4b^2) + (c^4 + 4c^2) + (d^4 + 4d^2) \\ &\geq 2\left(\sqrt{a^4 \cdot 4a^2} + \sqrt{b^4 \cdot 4b^2} + \sqrt{c^4 \cdot 4c^2} + \sqrt{d^4 \cdot 4d^2}\right) \\ &= 4(|a^3| + |b^3| + |c^3| + |d^3|) \geq 4(a^3 + b^3 + c^3 + d^3), \end{aligned}$$

which establishes the rightmost inequality.

To prove the leftmost inequality, we first show that $a, b, c, d \in [0,3]$ as in the previous solution. Moreover, we can assume that $0 \leq a \leq b \leq c \leq d$. Then we have $a + b \leq b + c \leq \frac{2}{3}(b+c+d) \leq \frac{2}{3} \cdot 6 = 4$.

Next, we show that $4b - b^2 \leq 4c - c^2$. Actually, this inequality rewrites as $(c-b)(b+c-4) \leq 0$, which follows from the previous estimate. The inequality $4a - a^2 \leq 4b - b^2$ can be proved analogously.

Further, the inequalities $a \leq b \leq c$ together with $4a - a^2 \leq 4b - b^2 \leq 4c - c^2$ allow us to apply the Chebyshev inequality obtaining

$$\begin{aligned} a^2(4a - a^2) + b^2(4b - b^2) + c^2(4c - c^2) &\geq \frac{1}{3}(a^2 + b^2 + c^2)\left(4(a+b+c) - (a^2 + b^2 + c^2)\right) \\ &= \frac{(12 - d^2)(4(6-d) - (12 - d^2))}{3}. \end{aligned}$$

This implies that

$$\begin{aligned} (4a^3 - a^4) + (4b^3 - b^4) + (4c^3 - c^4) + (4d^3 - d^4) &\geq \frac{(12 - d^2)(d^2 - 4d + 12)}{3} + 4d^3 - d^4 \\ = \frac{144 - 48d + 16d^3 - 4d^4}{3} &= 36 + \frac{4}{3}(3-d)(d-1)(d^2 - 3). \qquad (2) \end{aligned}$$

Finally, we have $d^2 \geq \frac{1}{4}(a^2 + b^2 + c^2 + d^2) = 3$ (which implies $d > 1$); so, the expression $\frac{4}{3}(3-d)(d-1)(d^2-3)$ in the right-hand part of (2) is nonnegative, and the desired inequality is proved.

Comment. The rightmost inequality is easier than the leftmost one. In particular, Solutions 2 and 3 show that only the condition $a^2 + b^2 + c^2 + d^2 = 12$ is needed for the former one.

A3. Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \le 1$ for all $i = 1, \ldots, 100$ (we put $x_{101} = x_1$, $x_{102} = x_2$). Find the maximal possible value of the sum

$$S = \sum_{i=1}^{100} x_i x_{i+2}.$$

(Russia)

Answer. $\dfrac{25}{2}$.

Solution 1. Let $x_{2i} = 0$, $x_{2i-1} = \frac{1}{2}$ for all $i = 1, \ldots, 50$. Then we have $S = 50 \cdot \left(\frac{1}{2}\right)^2 = \frac{25}{2}$. So, we are left to show that $S \le \frac{25}{2}$ for all values of x_i's satisfying the problem conditions.

Consider any $1 \le i \le 50$. By the problem condition, we get $x_{2i-1} \le 1 - x_{2i} - x_{2i+1}$ and $x_{2i+2} \le 1 - x_{2i} - x_{2i+1}$. Hence by the AM–GM inequality we get

$$\begin{aligned} x_{2i-1}x_{2i+1} + x_{2i}x_{2i+2} &\le (1 - x_{2i} - x_{2i+1})x_{2i+1} + x_{2i}(1 - x_{2i} - x_{2i+1}) \\ &= (x_{2i} + x_{2i+1})(1 - x_{2i} - x_{2i+1}) \le \left(\frac{(x_{2i} + x_{2i+1}) + (1 - x_{2i} - x_{2i+1})}{2}\right)^2 = \frac{1}{4}. \end{aligned}$$

Summing up these inequalities for $i = 1, 2, \ldots, 50$, we get the desired inequality

$$\sum_{i=1}^{50} (x_{2i-1}x_{2i+1} + x_{2i}x_{2i+2}) \le 50 \cdot \frac{1}{4} = \frac{25}{2}.$$

Comment. This solution shows that a bit more general fact holds. Namely, consider $2n$ nonnegative numbers $x_1, \ldots, x_{2n}$ in a row (with no cyclic notation) and suppose that $x_i + x_{i+1} + x_{i+2} \le 1$ for all $i = 1, 2, \ldots, 2n-2$. Then $\sum\limits_{i=1}^{2n-2} x_i x_{i+2} \le \dfrac{n-1}{4}$.

The proof is the same as above, though if might be easier to find it (for instance, applying induction). The original estimate can be obtained from this version by considering the sequence $x_1, x_2, \ldots, x_{100}, x_1, x_2$.

Solution 2. We present another proof of the estimate. From the problem condition, we get

$$\begin{aligned} S = \sum_{i=1}^{100} x_i x_{i+2} \le \sum_{i=1}^{100} x_i(1 - x_i - x_{i+1}) &= \sum_{i=1}^{100} x_i - \sum_{i=1}^{100} x_i^2 - \sum_{i=1}^{100} x_i x_{i+1} \\ &= \sum_{i=1}^{100} x_i - \frac{1}{2}\sum_{i=1}^{100} (x_i + x_{i+1})^2. \end{aligned}$$

By the AM–QM inequality, we have $\sum (x_i + x_{i+1})^2 \ge \frac{1}{100}\left(\sum (x_i + x_{i+1})\right)^2$, so

$$\begin{aligned} S \le \sum_{i=1}^{100} x_i - \frac{1}{200}\left(\sum_{i=1}^{100}(x_i + x_{i+1})\right)^2 &= \sum_{i=1}^{100} x_i - \frac{2}{100}\left(\sum_{i=1}^{100} x_i\right)^2 \\ &= \frac{2}{100}\left(\sum_{i=1}^{100} x_i\right)\left(\frac{100}{2} - \sum_{i=1}^{100} x_i\right). \end{aligned}$$

And finally, by the AM–GM inequality

$$S \le \frac{2}{100} \cdot \left(\frac{1}{2}\left(\sum_{i=1}^{100} x_i + \frac{100}{2} - \sum_{i=1}^{100} x_i\right)\right)^2 = \frac{2}{100} \cdot \left(\frac{100}{4}\right)^2 = \frac{25}{2}.$$

Comment. These solutions are not as easy as they may seem at the first sight. There are two different optimal configurations in which the variables have different values, and not all of sums of three consecutive numbers equal 1. Although it is easy to find the value $\frac{25}{2}$, the estimates must be done with care to preserve equality in the optimal configurations.

A4. A sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k$, $x_{2k-1} = (-1)^{k+1}x_k$ for all $k \geq 1$. Prove that $x_1 + x_2 + \cdots + x_n \geq 0$ for all $n \geq 1$.

(Austria)

Solution 1. We start with some observations. First, from the definition of x_i it follows that for each positive integer k we have

$$x_{4k-3} = x_{2k-1} = -x_{4k-2} \quad \text{and} \quad x_{4k-1} = x_{4k} = -x_{2k} = x_k. \tag{1}$$

Hence, denoting $S_n = \sum_{i=1}^n x_i$, we have

$$S_{4k} = \sum_{i=1}^{k}\big((x_{4k-3} + x_{4k-2}) + (x_{4k-1} + x_{4k})\big) = \sum_{i=1}^{k}(0 + 2x_k) = 2S_k, \tag{2}$$

$$S_{4k+2} = S_{4k} + (x_{4k+1} + x_{4k+2}) = S_{4k}. \tag{3}$$

Observe also that $S_n = \sum_{i=1}^n x_i \equiv \sum_{i=1}^n 1 = n \pmod 2$.

Now we prove by induction on k that $S_i \geq 0$ for all $i \leq 4k$. The base case is valid since $x_1 = x_3 = x_4 = 1$, $x_2 = -1$. For the induction step, assume that $S_i \geq 0$ for all $i \leq 4k$. Using the relations (1)–(3), we obtain

$$S_{4k+4} = 2S_{k+1} \geq 0, \quad S_{4k+2} = S_{4k} \geq 0, \quad S_{4k+3} = S_{4k+2} + x_{4k+3} = \frac{S_{4k+2} + S_{4k+4}}{2} \geq 0.$$

So, we are left to prove that $S_{4k+1} \geq 0$. If k is odd, then $S_{4k} = 2S_k \geq 0$; since k is odd, S_k is odd as well, so we have $S_{4k} \geq 2$ and hence $S_{4k+1} = S_{4k} + x_{4k+1} \geq 1$.

Conversely, if k is even, then we have $x_{4k+1} = x_{2k+1} = x_{k+1}$, hence $S_{4k+1} = S_{4k} + x_{4k+1} = 2S_k + x_{k+1} = S_k + S_{k+1} \geq 0$. The step is proved.

Solution 2. We will use the notation of S_n and the relations (1)–(3) from the previous solution.

Assume the contrary and consider the minimal n such that $S_{n+1} < 0$; surely $n \geq 1$, and from $S_n \geq 0$ we get $S_n = 0$, $x_{n+1} = -1$. Hence, we are especially interested in the set $M = \{n : S_n = 0\}$; our aim is to prove that $x_{n+1} = 1$ whenever $n \in M$ thus coming to a contradiction.

For this purpose, we first describe the set M inductively. We claim that (i) M consists only of even numbers, (ii) $2 \in M$, and (iii) for every even $n \geq 4$ we have $n \in M \iff [n/4] \in M$. Actually, (i) holds since $S_n \equiv n \pmod 2$, (ii) is straightforward, while (iii) follows from the relations $S_{4k+2} = S_{4k} = 2S_k$.

Now, we are left to prove that $x_{n+1} = 1$ if $n \in M$. We use the induction on n. The base case is $n = 2$, that is, the minimal element of M; here we have $x_3 = 1$, as desired.

For the induction step, consider some $4 \leq n \in M$ and let $m = [n/4] \in M$; then m is even, and $x_{m+1} = 1$ by the induction hypothesis. We prove that $x_{n+1} = x_{m+1} = 1$. If $n = 4m$ then we have $x_{n+1} = x_{2m+1} = x_{m+1}$ since m is even; otherwise, $n = 4m+2$, and $x_{n+1} = -x_{2m+2} = x_{m+1}$, as desired. The proof is complete.

Comment. Using the inductive definition of set M, one can describe it explicitly. Namely, M consists exactly of all positive integers not containing digits 1 and 3 in their 4-base representation.

A5. Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:

$$f\big(f(x)^2 y\big) = x^3 f(xy). \tag{1}$$

(Switzerland)

Answer. The only such function is $f(x) = \dfrac{1}{x}$.

Solution. By substituting $y = 1$, we get

$$f\big(f(x)^2\big) = x^3 f(x). \tag{2}$$

Then, whenever $f(x) = f(y)$, we have

$$x^3 = \frac{f\big(f(x)^2\big)}{f(x)} = \frac{f\big(f(y)^2\big)}{f(y)} = y^3$$

which implies $x = y$, so the function f is injective.

Now replace x by xy in (2), and apply (1) twice, second time to $\big(y, f(x)^2\big)$ instead of (x, y):

$$f\big(f(xy)^2\big) = (xy)^3 f(xy) = y^3 f\big(f(x)^2 y\big) = f\big(f(x)^2 f(y)^2\big).$$

Since f is injective, we get

$$f(xy)^2 = f(x)^2 f(y)^2,$$
$$f(xy) = f(x) f(y).$$

Therefore, f is multiplicative. This also implies $f(1) = 1$ and $f(x^n) = f(x)^n$ for all integers n.

Then the function equation (1) can be re-written as

$$f\big(f(x)\big)^2 f(y) = x^3 f(x) f(y),$$
$$f\big(f(x)\big) = \sqrt{x^3 f(x)}. \tag{3}$$

Let $g(x) = x f(x)$. Then, by (3), we have

$$g\big(g(x)\big) = g\big(x f(x)\big) = x f(x) \cdot f\big(x f(x)\big) = x f(x)^2 f\big(f(x)\big) =$$
$$= x f(x)^2 \sqrt{x^3 f(x)} = \big(x f(x)\big)^{5/2} = \big(g(x)\big)^{5/2},$$

and, by induction,

$$\underbrace{g\Big(g\big(\ldots g}_{n+1}(x)\ldots\big)\Big) = \big(g(x)\big)^{(5/2)^n} \tag{4}$$

for every positive integer n.

Consider (4) for a fixed x. The left-hand side is always rational, so $\big(g(x)\big)^{(5/2)^n}$ must be rational for every n. We show that this is possible only if $g(x) = 1$. Suppose that $g(x) \neq 1$, and let the prime factorization of $g(x)$ be $g(x) = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ where $p_1, \ldots, p_k$ are distinct primes and $\alpha_1, \ldots, \alpha_k$ are nonzero integers. Then the unique prime factorization of (4) is

$$\underbrace{g\Big(g\big(\ldots g}_{n+1}(x)\ldots\big)\Big) = \big(g(x)\big)^{(5/2)^n} = p_1^{(5/2)^n \alpha_1} \cdots p_k^{(5/2)^n \alpha_k}$$

where the exponents should be integers. But this is not true for large values of n, for example $(\frac{5}{2})^n\alpha_1$ cannot be a integer number when $2^n \not| \alpha_1$. Therefore, $g(x) \neq 1$ is impossible.

Hence, $g(x) = 1$ and thus $f(x) = \dfrac{1}{x}$ for all x.

The function $f(x) = \dfrac{1}{x}$ satisfies the equation (1):

$$f(f(x)^2y) = \frac{1}{f(x)^2y} = \frac{1}{\left(\frac{1}{x}\right)^2 y} = \frac{x^3}{xy} = x^3 f(xy).$$

Comment. Among $\mathbb{R}^+ \to \mathbb{R}^+$ functions, $f(x) = \dfrac{1}{x}$ is not the only solution. Another solution is $f_1(x) = x^{3/2}$. Using transfinite tools, infinitely many other solutions can be constructed.

A6. Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer n.

(Germany)

Solution 1. Throughout the solution, by $\mathbb{N}$ we denote the set of all positive integers. For any function $h\colon \mathbb{N} \to \mathbb{N}$ and for any positive integer k, define $h^k(x) = \underbrace{h\bigl(h\bigl(\dots h}_{k}(x)\dots\bigr)\bigr)$ (in particular, $h^0(x) = x$).

Observe that $f\bigl(g^k(x)\bigr) = f\bigl(g^{k-1}(x)\bigr) + 1 = \dots = f(x) + k$ for any positive integer k, and similarly $g\bigl(f^k(x)\bigr) = g(x) + k$. Now let a and b are the minimal values attained by f and g, respectively; say $f(n_f) = a$, $g(n_g) = b$. Then we have $f\bigl(g^k(n_f)\bigr) = a + k$, $g\bigl(f^k(n_g)\bigr) = b + k$, so the function f attains all values from the set $N_f = \{a, a+1, \dots\}$, while g attains all the values from the set $N_g = \{b, b+1, \dots\}$.

Next, note that $f(x) = f(y)$ implies $g(x) = g\bigl(f(x)\bigr) - 1 = g\bigl(f(y)\bigr) - 1 = g(y)$; surely, the converse implication also holds. Now, we say that x and y are *similar* (and write $x \sim y$) if $f(x) = f(y)$ (equivalently, $g(x) = g(y)$). For every $x \in \mathbb{N}$, we define $[x] = \{y \in \mathbb{N} : x \sim y\}$; surely, $y_1 \sim y_2$ for all $y_1, y_2 \in [x]$, so $[x] = [y]$ whenever $y \in [x]$.

Now we investigate the structure of the sets $[x]$.

Claim 1. Suppose that $f(x) \sim f(y)$; then $x \sim y$, that is, $f(x) = f(y)$. Consequently, each class $[x]$ contains at most one element from N_f, as well as at most one element from N_g.

Proof. If $f(x) \sim f(y)$, then we have $g(x) = g\bigl(f(x)\bigr) - 1 = g\bigl(f(y)\bigr) - 1 = g(y)$, so $x \sim y$. The second statement follows now from the sets of values of f and g. $\square$

Next, we clarify which classes do not contain large elements.

Claim 2. For any $x \in \mathbb{N}$, we have $[x] \subseteq \{1, 2, \dots, b-1\}$ if and only if $f(x) = a$. Analogously, $[x] \subseteq \{1, 2, \dots, a-1\}$ if and only if $g(x) = b$.

Proof. We will prove that $[x] \nsubseteq \{1, 2, \dots, b-1\} \iff f(x) > a$; the proof of the second statement is similar.

Note that $f(x) > a$ implies that there exists some y satisfying $f(y) = f(x) - 1$, so $f\bigl(g(y)\bigr) = f(y) + 1 = f(x)$, and hence $x \sim g(y) \geq b$. Conversely, if $b \leq c \sim x$ then $c = g(y)$ for some $y \in \mathbb{N}$, which in turn follows $f(x) = f\bigl(g(y)\bigr) = f(y) + 1 \geq a + 1$, and hence $f(x) > a$. $\square$

Claim 2 implies that there exists exactly one class contained in $\{1, \dots, a-1\}$ (that is, the class $[n_g]$), as well as exactly one class contained in $\{1, \dots, b-1\}$ (the class $[n_f]$). Assume for a moment that $a \leq b$; then $[n_g]$ is contained in $\{1, \dots, b-1\}$ as well, hence it coincides with $[n_g]$. So, we get that

$$f(x) = a \iff g(x) = b \iff x \sim n_f \sim n_g. \tag{1}$$

Claim 3. $a = b$.

Proof. By Claim 2, we have $[a] \neq [n_f]$, so $[a]$ should contain some element $a' \geq b$ by Claim 2 again. If $a \neq a'$, then $[a]$ contains two elements $\geq a$ which is impossible by Claim 1. Therefore, $a = a' \geq b$. Similarly, $b \geq a$. $\square$

Now we are ready to prove the problem statement. First, we establish the following

Claim 4. For every integer $d \geq 0$, $f^{d+1}(n_f) = g^{d+1}(n_f) = a + d$.

Proof. Induction on d. For $d = 0$, the statement follows from (1) and Claim 3. Next, for $d > 1$ from the induction hypothesis we have $f^{d+1}(n_f) = f\bigl(f^d(n_f)\bigr) = f\bigl(g^d(n_f)\bigr) = f(n_f) + d = a + d$. The equality $g^{d+1}(n_f) = a + d$ is analogous. $\square$

Finally, for each $x \in \mathbb{N}$, we have $f(x) = a + d$ for some $d \geq 0$, so $f(x) = f\big(g^d(n_f)\big)$ and hence $x \sim g^d(n_f)$. It follows that $g(x) = g\big(g^d(n_f)\big) = g^{d+1}(n_f) = a + d = f(x)$ by Claim 4.

Solution 2. We start with the same observations, introducing the relation $\sim$ and proving Claim 1 from the previous solution.

Note that $f(a) > a$ since otherwise we have $f(a) = a$ and hence $g(a) = g\big(f(a)\big) = g(a) + 1$, which is false.

Claim 2′. $a = b$.

Proof. We can assume that $a \leq b$. Since $f(a) \geq a + 1$, there exists some $x \in \mathbb{N}$ such that $f(a) = f(x) + 1$, which is equivalent to $f(a) = f\big(g(x)\big)$ and $a \sim g(x)$. Since $g(x) \geq b \geq a$, by Claim 1 we have $a = g(x) \geq b$, which together with $a \leq b$ proves the Claim. □

Now, almost the same method allows to find the values $f(a)$ and $g(a)$.

Claim 3′. $f(a) = g(a) = a + 1$.

Proof. Assume the contrary; then $f(a) \geq a + 2$, hence there exist some $x, y \in \mathbb{N}$ such that $f(x) = f(a) - 2$ and $f(y) = g(x)$ (as $g(x) \geq a = b$). Now we get $f(a) = f(x) + 2 = f\big(g^2(x)\big)$, so $a \sim g^2(x) \geq a$, and by Claim 1 we get $a = g^2(x) = g\big(f(y)\big) = 1 + g(y) \geq 1 + a$; this is impossible. The equality $g(a) = a + 1$ is similar.

Now, we are prepared for the proof of the problem statement. First, we prove it for $n \geq a$.

Claim 4′. For each integer $x \geq a$, we have $f(x) = g(x) = x + 1$.

Proof. Induction on x. The base case $x = a$ is provided by Claim 3′, while the induction step follows from $f(x+1) = f\big(g(x)\big) = f(x) + 1 = (x+1) + 1$ and the similar computation for $g(x+1)$.

Finally, for an arbitrary $n \in \mathbb{N}$ we have $g(n) \geq a$, so by Claim 4′ we have $f(n) + 1 = f\big(g(n)\big) = g(n) + 1$, hence $f(n) = g(n)$.

Comment. It is not hard now to describe all the functions $f : \mathbb{N} \to \mathbb{N}$ satisfying the property $f(f(n)) = f(n) + 1$. For each such function, there exists $n_0 \in \mathbb{N}$ such that $f(n) = n + 1$ for all $n \geq n_0$, while for each $n < n_0$, $f(n)$ is an arbitrary number greater than of equal to n_0 (these numbers may be different for different $n < n_0$).

A7. Let $a_1, \ldots, a_r$ be positive real numbers. For $n > r$, we inductively define

$$a_n = \max_{1 \le k \le n-1} (a_k + a_{n-k}). \tag{1}$$

Prove that there exist positive integers $\ell \le r$ and N such that $a_n = a_{n-\ell} + a_\ell$ for all $n \ge N$.

(Iran)

Solution 1. First, from the problem conditions we have that each a_n $(n > r)$ can be expressed as $a_n = a_{j_1} + a_{j_2}$ with $j_1, j_2 < n$, $j_1 + j_2 = n$. If, say, $j_1 > r$ then we can proceed in the same way with a_{j_1}, and so on. Finally, we represent a_n in a form

$$a_n = a_{i_1} + \cdots + a_{i_k}, \tag{2}$$

$$1 \le i_j \le r, \quad i_1 + \cdots + i_k = n. \tag{3}$$

Moreover, if a_{i_1} and a_{i_2} are the numbers in (2) obtained on the last step, then $i_1 + i_2 > r$. Hence we can adjust (3) as

$$1 \le i_j \le r, \quad i_1 + \cdots + i_k = n, \quad i_1 + i_2 > r. \tag{4}$$

On the other hand, suppose that the indices $i_1, \ldots, i_k$ satisfy the conditions (4). Then, denoting $s_j = i_1 + \cdots + i_j$, from (1) we have

$$a_n = a_{s_k} \ge a_{s_{k-1}} + a_{i_k} \ge a_{s_{k-2}} + a_{i_{k-1}} + a_{i_k} \ge \cdots \ge a_{i_1} + \cdots + a_{i_k}.$$

Summarizing these observations we get the following

Claim. For every $n > r$, we have

$$a_n = \max\{a_{i_1} + \cdots + a_{i_k} : \text{the collection } (i_1, \ldots, i_k) \text{ satisfies (4)}\}. \qquad \square$$

Now we denote

$$s = \max_{1 \le i \le r} \frac{a_i}{i}$$

and fix some index $\ell \le r$ such that $s = \dfrac{a_\ell}{\ell}$.

Consider some $n \ge r^2\ell + 2r$ and choose an expansion of a_n in the form (2), (4). Then we have $n = i_1 + \cdots + i_k \le rk$, so $k \ge n/r \ge r\ell + 2$. Suppose that none of the numbers $i_3, \ldots, i_k$ equals ℓ. Then by the pigeonhole principle there is an index $1 \le j \le r$ which appears among $i_3, \ldots, i_k$ at least ℓ times, and surely $j \ne \ell$. Let us delete these ℓ occurrences of j from $(i_1, \ldots, i_k)$, and add j occurrences of ℓ instead, obtaining a sequence $(i_1, i_2, i'_3, \ldots, i'_{k'})$ also satisfying (4). By Claim, we have

$$a_{i_1} + \cdots + a_{i_k} = a_n \ge a_{i_1} + a_{i_2} + a_{i'_3} + \cdots + a_{i'_{k'}},$$

or, after removing the coinciding terms, $\ell a_j \ge j a_\ell$, so $\dfrac{a_\ell}{\ell} \le \dfrac{a_j}{j}$. By the definition of ℓ, this means that $\ell a_j = j a_\ell$, hence

$$a_n = a_{i_1} + a_{i_2} + a_{i'_3} + \cdots + a_{i'_{k'}}.$$

Thus, for every $n \ge r^2\ell + 2r$ we have found a representation of the form (2), (4) with $i_j = \ell$ for some $j \ge 3$. Rearranging the indices we may assume that $i_k = \ell$.

Finally, observe that in this representation, the indices $(i_1, \ldots, i_{k-1})$ satisfy the conditions (4) with n replaced by $n - \ell$. Thus, from the Claim we get

$$a_{n-\ell} + a_\ell \ge (a_{i_1} + \cdots + a_{i_{k-1}}) + a_\ell = a_n,$$

which by (1) implies

$$a_n = a_{n-\ell} + a_\ell \qquad \text{for each } n \ge r^2\ell + 2r,$$

as desired.

Solution 2. As in the previous solution, we involve the expansion (2), (3), and we fix some index $1 \le \ell \le r$ such that
$$\frac{a_\ell}{\ell} = s = \max_{1\le i\le r} \frac{a_i}{i}.$$
Now, we introduce the sequence (b_n) as $b_n = a_n - sn$; then $b_\ell = 0$.

We prove by induction on n that $b_n \le 0$, and (b_n) satisfies the same recurrence relation as (a_n). The base cases $n \le r$ follow from the definition of s. Now, for $n > r$ from the induction hypothesis we have

$$b_n = \max_{1\le k\le n-1}(a_k + a_{n-k}) - ns = \max_{1\le k\le n-1}(b_k + b_{n-k} + ns) - ns = \max_{1\le k\le n-1}(b_k + b_{n-k}) \le 0,$$

as required.

Now, if $b_k = 0$ for all $1 \le k \le r$, then $b_n = 0$ for all n, hence $a_n = sn$, and the statement is trivial. Otherwise, define

$$M = \max_{1\le i\le r} |b_i|, \quad \varepsilon = \min\{|b_i| : 1 \le i \le r,\ b_i < 0\}.$$

Then for $n > r$ we obtain

$$b_n = \max_{1\le k\le n-1}(b_k + b_{n-k}) \ge b_\ell + b_{n-\ell} = b_{n-\ell},$$

so
$$0 \ge b_n \ge b_{n-\ell} \ge b_{n-2\ell} \ge \cdots \ge -M.$$

Thus, in view of the expansion (2), (3) applied to the sequence (b_n), we get that each b_n is contained in a set

$$T = \{b_{i_1} + b_{i_2} + \cdots + b_{i_k} :\ i_1, \ldots, i_k \le r\} \cap [-M, 0]$$

We claim that this set is finite. Actually, for any $x \in T$, let $x = b_{i_1} + \cdots + b_{i_k}$ $(i_1, \ldots, i_k \le r)$. Then among b_{i_j}'s there are at most $\frac{M}{\varepsilon}$ nonzero terms (otherwise $x < \frac{M}{\varepsilon} \cdot (-\varepsilon) < -M$). Thus x can be expressed in the same way with $k \le \frac{M}{\varepsilon}$, and there is only a finite number of such sums.

Finally, for every $t = 1, 2, \ldots, \ell$ we get that the sequence

$$b_{r+t}, b_{r+t+\ell}, b_{r+t+2\ell}, \ldots$$

is non-decreasing and attains the finite number of values; therefore it is constant from some index. Thus, the sequence (b_n) is periodic with period ℓ from some index N, which means that

$$b_n = b_{n-\ell} = b_{n-\ell} + b_\ell \qquad \text{for all } n > N + \ell,$$

and hence

$$a_n = b_n + ns = (b_{n-\ell} + (n-\ell)s) + (b_\ell + \ell s) = a_{n-\ell} + a_\ell \qquad \text{for all } n > N + \ell,$$

as desired.

A8. Given six positive numbers a, b, c, d, e, f such that $a < b < c < d < e < f$. Let $a+c+e = S$ and $b + d + f = T$. Prove that

$$2ST > \sqrt{3(S+T)\big(S(bd+bf+df)+T(ac+ae+ce)\big)}. \qquad (1)$$

(South Korea)

Solution 1. We define also $\sigma = ac + ce + ae$, $\tau = bd + bf + df$. The idea of the solution is to interpret (1) as a natural inequality on the roots of an appropriate polynomial.

Actually, consider the polynomial

$$\begin{aligned} P(x) &= (b+d+f)(x-a)(x-c)(x-e) + (a+c+e)(x-b)(x-d)(x-f) \\ &= T(x^3 - Sx^2 + \sigma x - ace) + S(x^3 - Tx^2 + \tau x - bdf). \qquad (2) \end{aligned}$$

Surely, P is cubic with leading coefficient $S + T > 0$. Moreover, we have

$$P(a) = S(a-b)(a-d)(a-f) < 0, \qquad P(c) = S(c-b)(c-d)(c-f) > 0,$$
$$P(e) = S(e-b)(e-d)(e-f) < 0, \qquad P(f) = T(f-a)(f-c)(f-e) > 0.$$

Hence, each of the intervals (a, c), (c, e), (e, f) contains at least one root of $P(x)$. Since there are at most three roots at all, we obtain that there is exactly one root in each interval (denote them by $\alpha \in (a, c)$, $\beta \in (c, e)$, $\gamma \in (e, f)$). Moreover, the polynomial P can be factorized as

$$P(x) = (T+S)(x-\alpha)(x-\beta)(x-\gamma). \qquad (3)$$

Equating the coefficients in the two representations (2) and (3) of $P(x)$ provides

$$\alpha + \beta + \gamma = \frac{2TS}{T+S}, \qquad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{S\tau + T\sigma}{T+S}.$$

Now, since the numbers α, β, γ are distinct, we have

$$0 < (\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\beta-\gamma)^2 = 2(\alpha+\beta+\gamma)^2 - 6(\alpha\beta+\alpha\gamma+\beta\gamma),$$

which implies

$$\frac{4S^2T^2}{(T+S)^2} = (\alpha+\beta+\gamma)^2 > 3(\alpha\beta+\alpha\gamma+\beta\gamma) = \frac{3(S\tau+T\sigma)}{T+S},$$

or

$$4S^2T^2 > 3(T+S)(T\sigma + S\tau),$$

which is exactly what we need.

Comment 1. In fact, one can locate the roots of $P(x)$ more narrowly: they should lie in the intervals (a, b), (c, d), (e, f).

Surely, if we change all inequality signs in the problem statement to non-strict ones, the (non-strict) inequality will also hold by continuity. One can also find when the equality is achieved. This happens in that case when $P(x)$ is a perfect cube, which immediately implies that $b = c = d = e (= \alpha = \beta = \gamma)$, together with the additional condition that $P''(b) = 0$. Algebraically,

$$\begin{aligned} 6(T+S)b - 4TS = 0 \quad &\iff \quad 3b(a+4b+f) = 2(a+2b)(2b+f) \\ &\iff \quad f = \frac{b(4b-a)}{2a+b} = b\left(1 + \frac{3(b-a)}{2a+b}\right) > b. \end{aligned}$$

This means that for every pair of numbers a, b such that $0 < a < b$, there exists $f > b$ such that the point (a, b, b, b, b, f) is a point of equality.

Solution 2. Let
$$U=\frac{1}{2}\big((e-a)^2+(c-a)^2+(e-c)^2\big)=S^2-3(ac+ae+ce)$$
and
$$V=\frac{1}{2}\big((f-b)^2+(f-d)^2+(d-b)^2\big)=T^2-3(bd+bf+df).$$
Then
$$(\text{L.H.S.})^2-(\text{R.H.S.})^2=(2ST)^2-(S+T)\big(S\cdot 3(bd+bf+df)+T\cdot 3(ac+ae+ce)\big)=$$
$$=4S^2T^2-(S+T)\big(S(T^2-V)+T(S^2-U)\big)=(S+T)(SV+TU)-ST(T-S)^2,$$
and the statement is equivalent with
$$(S+T)(SV+TU)>ST(T-S)^2. \tag{4}$$

By the Cauchy-Schwarz inequality,
$$(S+T)(TU+SV)\geq\big(\sqrt{S\cdot TU}+\sqrt{T\cdot SV}\big)^2=ST\big(\sqrt{U}+\sqrt{V}\big)^2. \tag{5}$$
Estimate the quantities $\sqrt{U}$ and $\sqrt{V}$ by the QM–AM inequality with the positive terms $(e-c)^2$ and $(d-b)^2$ being omitted:
$$\begin{aligned}\sqrt{U}+\sqrt{V}&>\sqrt{\frac{(e-a)^2+(c-a)^2}{2}}+\sqrt{\frac{(f-b)^2+(f-d)^2}{2}}\\&>\frac{(e-a)+(c-a)}{2}+\frac{(f-b)+(f-d)}{2}=\left(f-\frac{d}{2}-\frac{b}{2}\right)+\left(\frac{e}{2}+\frac{c}{2}-a\right)\\&=(T-S)+\frac{3}{2}(e-d)+\frac{3}{2}(c-b)>T-S.\end{aligned} \tag{6}$$

The estimates (5) and (6) prove (4) and hence the statement.

Solution 3. We keep using the notations σ and τ from Solution 1. Moreover, let $s=c+e$. Note that
$$(c-b)(c-d)+(e-f)(e-d)+(e-f)(c-b)<0,$$
since each summand is negative. This rewrites as
$$(bd+bf+df)-(ac+ce+ae)<(c+e)(b+d+f-a-c-e),\ \text{or}$$
$$\tau-\sigma<s(T-S). \tag{7}$$
Then we have
$$\begin{aligned}S\tau+T\sigma&=S(\tau-\sigma)+(S+T)\sigma<Ss(T-S)+(S+T)(ce+as)\\&\leq Ss(T-S)+(S+T)\left(\frac{s^2}{4}+(S-s)s\right)=s\left(2ST-\frac{3}{4}(S+T)s\right).\end{aligned}$$
Using this inequality together with the AM–GM inequality we get
$$\begin{aligned}\sqrt{\frac{3}{4}(S+T)(S\tau+T\sigma)}&<\sqrt{\frac{3}{4}(S+T)s\left(2ST-\frac{3}{4}(S+T)s\right)}\\&\leq\frac{\frac{3}{4}(S+T)s+2ST-\frac{3}{4}(S+T)s}{2}=ST.\end{aligned}$$
Hence,
$$2ST>\sqrt{3(S+T)\big(S(bd+bf+df)+T(ac+ae+ce)\big)}.$$

Comment 2. The expression (7) can be found by considering the sum of the roots of the quadratic polynomial $q(x) = (x-b)(x-d)(x-f) - (x-a)(x-c)(x-e)$.

Solution 4. We introduce the expressions σ and τ as in the previous solutions. The idea of the solution is to change the values of variables $a, \ldots, f$ keeping the left-hand side unchanged and increasing the right-hand side; it will lead to a simpler inequality which can be proved in a direct way.

Namely, we change the variables (i) keeping the (non-strict) inequalities $a \le b \le c \le d \le e \le f$; (ii) keeping the values of sums S and T unchanged; and finally (iii) increasing the values of σ and τ. Then the left-hand side of (1) remains unchanged, while the right-hand side increases. Hence, the inequality (1) (and even a non-strict version of (1)) for the changed values would imply the same (strict) inequality for the original values.

First, we find the sufficient conditions for (ii) and (iii) to be satisfied.

Lemma. Let $x, y, z > 0$; denote $U(x, y, z) = x + y + z$, $\upsilon(x, y, z) = xy + xz + yz$. Suppose that $x' + y' = x + y$ but $|x - y| \ge |x' - y'|$; then we have $U(x', y', z) = U(x, y, z)$ and $\upsilon(x', y', z) \ge \upsilon(x, y, z)$ with equality achieved only when $|x - y| = |x' - y'|$.

Proof. The first equality is obvious. For the second, we have

$$\begin{aligned}\upsilon(x', y', z) = z(x' + y') + x'y' = z(x' + y') + \frac{(x' + y')^2 - (x' - y')^2}{4} \\ \ge z(x + y) + \frac{(x + y)^2 - (x - y)^2}{4} = \upsilon(x, y, z),\end{aligned}$$

with the equality achieved only for $(x' - y')^2 = (x - y)^2 \iff |x' - y'| = |x - y|$, as desired. □

Now, we apply Lemma several times making the following changes. For each change, we denote the new values by the same letters to avoid cumbersome notations.

1. Let $k = \dfrac{d-c}{2}$. Replace (b, c, d, e) by $(b+k, c+k, d-k, e-k)$. After the change we have $a < b < c = d < e < f$, the values of S, T remain unchanged, but σ, τ strictly increase by Lemma.

2. Let $\ell = \dfrac{e-d}{2}$. Replace (c, d, e, f) by $(c+\ell, d+\ell, e-\ell, f-\ell)$. After the change we have $a < b < c = d = e < f$, the values of S, T remain unchanged, but σ, τ strictly increase by the Lemma.

3. Finally, let $m = \dfrac{c-b}{3}$. Replace (a, b, c, d, e, f) by $(a+2m, b+2m, c-m, d-m, e-m, f-m)$. After the change, we have $a < b = c = d = e < f$ and S, T are unchanged. To check (iii), we observe that our change can be considered as a composition of two changes: $(a, b, c, d) \to (a+m, b+m, c-m, d-m)$ and $(a, b, e, f) \to (a+m, b+m, e-m, f-m)$. It is easy to see that each of these two consecutive changes satisfy the conditions of the Lemma, hence the values of σ and τ increase.

Finally, we come to the situation when $a < b = c = d = e < f$, and we need to prove the inequality

$$\begin{aligned}2(a + 2b)(2b + f) &\ge \sqrt{3(a + 4b + f)\big((a + 2b)(b^2 + 2bf) + (2b + f)(2ab + b^2)\big)} \\ &= \sqrt{3b(a + 4b + f) \cdot \big((a + 2b)(b + 2f) + (2b + f)(2a + b)\big)}. \qquad (8)\end{aligned}$$

Now, observe that

$$2 \cdot 2(a + 2b)(2b + f) = 3b(a + 4b + f) + \big((a + 2b)(b + 2f) + (2a + b)(2b + f)\big).$$

Hence (4) rewrites as

$$3b(a+4b+f)+\big((a+2b)(b+2f)+(2a+b)(2b+f)\big)$$
$$\geq 2\sqrt{3b(a+4b+f)\cdot\big((a+2b)(b+2f)+(2b+f)(2a+b)\big)},$$

which is simply the AM–GM inequality.

Comment 3. Here, we also can find all the cases of equality. Actually, it is easy to see that if some two numbers among b, c, d, e are distinct then one can use Lemma to increase the right-hand side of (1). Further, if $b=c=d=e$, then we need equality in (4); this means that we apply AM–GM to equal numbers, that is,

$$3b(a+4b+f)=(a+2b)(b+2f)+(2a+b)(2b+f),$$

which leads to the same equality as in Comment 1.

Combinatorics

C1. In a concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

(Austria)

Answer. Yes, such an example exists.

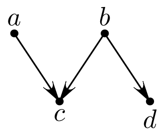
Solution. We say that an order of singers is *good* if it satisfied all their wishes. Next, we say that a number N is *realizable* by k singers (or k-*realizable*) if for some set of wishes of these singers there are exactly N good orders. Thus, we have to prove that a number 2010 is 20-realizable.

We start with the following simple

Lemma. Suppose that numbers n_1, n_2 are realizable by respectively k_1 and k_2 singers. Then the number $n_1 n_2$ is $(k_1 + k_2)$-realizable.

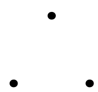
Proof. Let the singers $A_1, \ldots, A_{k_1}$ (with some wishes among them) realize n_1, and the singers B_1, $\ldots$, B_{k_2} (with some wishes among them) realize n_2. Add to each singer B_i the wish to perform later than all the singers A_j. Then, each good order of the obtained set of singers has the form $(A_{i_1}, \ldots, A_{i_{k_1}}, B_{j_1}, \ldots, B_{j_{k_2}})$, where $(A_{i_1}, \ldots, A_{i_{k_1}})$ is a good order for A_i's and $(B_{j_1}, \ldots, B_{j_{k_2}})$ is a good order for B_j's. Conversely, each order of this form is obviously good. Hence, the number of good orders is $n_1 n_2$. $\square$

In view of Lemma, we show how to construct sets of singers containing 4, 3 and 13 singers and realizing the numbers 5, 6 and 67, respectively. Thus the number $2010 = 6 \cdot 5 \cdot 67$ will be realizable by $4 + 3 + 13 = 20$ singers. These companies of singers are shown in Figs. 1–3; the wishes are denoted by arrows, and the number of good orders for each Figure stands below in the brackets.



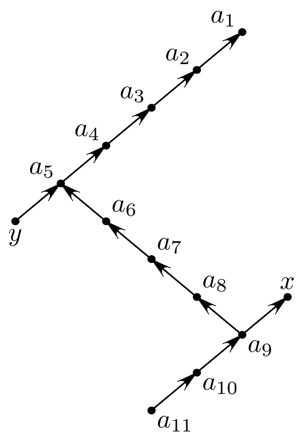
(5)

Fig. 1



(3)

Fig. 2



(67)

Fig. 3

For Fig. 1, there are exactly 5 good orders (a, b, c, d), (a, b, d, c), (b, a, c, d), (b, a, d, c), (b, d, a, c). For Fig. 2, each of 6 orders is good since there are no wishes.

Finally, for Fig. 3, the order of $a_1, \dots, a_{11}$ is fixed; in this line, singer x can stand before each of a_i ($i \le 9$), and singer y can stand after each of a_j ($j \ge 5$), thus resulting in $9 \cdot 7 = 63$ cases. Further, the positions of x and y in this line determine the whole order uniquely unless both of them come between the same pair (a_i, a_{i+1}) (thus $5 \le i \le 8$); in the latter cases, there are two orders instead of 1 due to the order of x and y. Hence, the total number of good orders is $63 + 4 = 67$, as desired.

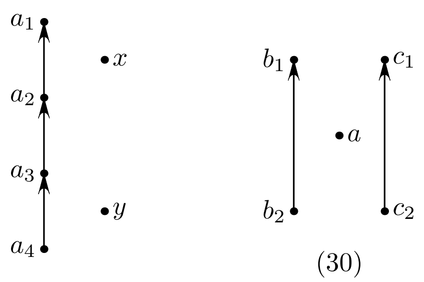
Comment. The number 20 in the problem statement is not sharp and is put there to respect the original formulation. So, if necessary, the difficulty level of this problem may be adjusted by replacing 20 by a smaller number. Here we present some improvements of the example leading to a smaller number of singers.

Surely, each example with < 20 singers can be filled with some "super-stars" who should perform at the very end in a fixed order. Hence each of these improvements provides a different solution for the problem. Moreover, the large variety of ideas standing behind these examples allows to suggest that there are many other examples.

1. Instead of building the examples realizing 5 and 6, it is more economic to make an example realizing 30; it may seem even simpler. Two possible examples consisting of 5 and 6 singers are shown in Fig. 4; hence the number 20 can be decreased to 19 or 18.

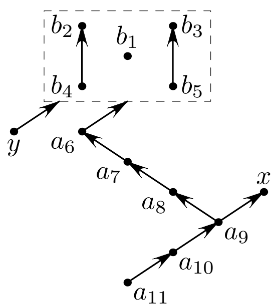
For Fig. 4a, the order of $a_1, \dots, a_4$ is fixed, there are 5 ways to add x into this order, and there are 6 ways to add y into the resulting order of $a_1, \dots, a_4, x$. Hence there are $5 \cdot 6 = 30$ good orders.

On Fig. 4b, for 5 singers a, b_1, b_2, c_1, c_2 there are $5! = 120$ orders at all. Obviously, exactly one half of them satisfies the wish $b_1 \leftarrow b_2$, and exactly one half of these orders satisfies another wish $c_1 \leftarrow c_2$; hence, there are exactly $5!/4 = 30$ good orders.



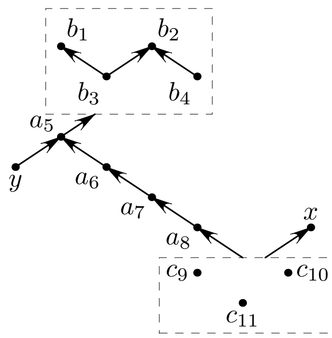
(30) (30)
a) b)

Fig. 4



(2010)

Fig. 5



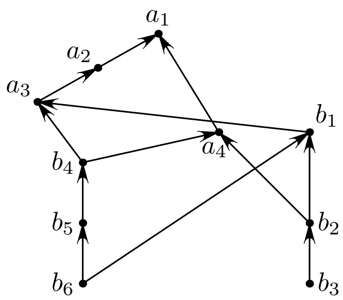
(2010)

Fig. 6

2. One can merge the examples for 30 and 67 shown in Figs. 4b and 3 in a smarter way, obtaining a set of 13 singers representing 2010. This example is shown in Fig. 5; an arrow from/to group $\{b_1, \dots, b_5\}$ means that there exists such arrow from each member of this group.

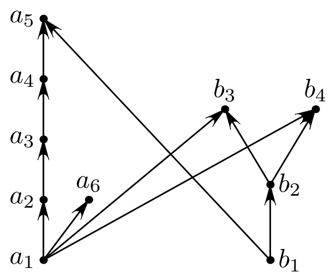
Here, as in Fig. 4b, one can see that there are exactly 30 orders of $b_1, \dots, b_5, a_6, \dots, a_{11}$ satisfying all their wishes among themselves. Moreover, one can prove in the same way as for Fig. 3 that each of these orders can be complemented by x and y in exactly 67 ways, hence obtaining $30 \cdot 67 = 2010$ good orders at all.

Analogously, one can merge the examples in Figs. 1–3 to represent 2010 by 13 singers as is shown in Fig. 6.



(67)

Fig. 7



(2010)

Fig. 8

3. Finally, we will present two other improvements; the proofs are left to the reader. The graph in Fig. 7 shows how 10 singers can represent 67. Moreover, even a company of 10 singers representing 2010 can be found; this company is shown in Fig. 8.

C2. On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag.

We say that a set of N flags is *diverse* if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

(Croatia)

Answer. $M = 2^{N-2} + 1$.

Solution. When speaking about the diagonal of a square, we will always mean the main diagonal.

Let M_N be the smallest positive integer satisfying the problem condition. First, we show that $M_N > 2^{N-2}$. Consider the collection of all 2^{N-2} flags having yellow first squares and blue second ones. Obviously, both colors appear on the diagonal of each $N \times N$ square formed by these flags.

We are left to show that $M_N \leq 2^{N-2} + 1$, thus obtaining the desired answer. We start with establishing this statement for $N = 4$.

Suppose that we have 5 flags of length 4. We decompose each flag into two parts of 2 squares each; thereby, we denote each flag as LR, where the 2×1 flags $L, R \in \mathcal{S} = \{\mathrm{BB}, \mathrm{BY}, \mathrm{YB}, \mathrm{YY}\}$ are its left and right parts, respectively. First, we make two easy observations on the flags 2×1 which can be checked manually.

(i) For each $A \in \mathcal{S}$, there exists only one 2×1 flag $C \in \mathcal{S}$ (possibly $C = A$) such that A and C cannot form a 2×2 square with monochrome diagonal (for part BB, that is YY, and for BY that is YB).

(ii) Let $A_1, A_2, A_3 \in \mathcal{S}$ be three distinct elements; then two of them can form a 2×2 square with yellow diagonal, and two of them can form a 2×2 square with blue diagonal (for all parts but BB, a pair (BY, YB) fits for both statements, while for all parts but BY, these pairs are (YB, YY) and (BB, YB)).

Now, let ℓ and r be the numbers of distinct left and right parts of our 5 flags, respectively. The total number of flags is $5 \leq r\ell$, hence one of the factors (say, r) should be at least 3. On the other hand, $\ell, r \leq 4$, so there are two flags with coinciding right part; let them be L_1R_1 and L_2R_1 ($L_1 \neq L_2$).

Next, since $r \geq 3$, there exist some flags L_3R_3 and L_4R_4 such that R_1, R_3, R_4 are distinct. Let $L'R'$ be the remaining flag. By (i), one of the pairs (L', L_1) and (L', L_2) can form a 2×2 square with monochrome diagonal; we can assume that L', L_2 form a square with a blue diagonal. Finally, the right parts of two of the flags L_1R_1, L_3R_3, L_4R_4 can also form a 2×2 square with a blue diagonal by (ii). Putting these 2×2 squares on the diagonal of a 4×4 square, we find a desired arrangement of four flags.

We are ready to prove the problem statement by induction on N; actually, above we have proved the base case $N = 4$. For the induction step, assume that $N > 4$, consider any $2^{N-2} + 1$ flags of length N, and arrange them into a large flag of size $(2^{N-2} + 1) \times N$. This flag contains a non-monochrome column since the flags are distinct; we may assume that this column is the first one. By the pigeonhole principle, this column contains at least $\left\lceil \dfrac{2^{N-2}+1}{2} \right\rceil = 2^{N-3} + 1$ squares of one color (say, blue). We call the flags with a blue first square *good*.

Consider all the good flags and remove the first square from each of them. We obtain at least $2^{N-3} + 1 \geq M_{N-1}$ flags of length $N - 1$; by the induction hypothesis, $N - 1$ of them can form a square Q with the monochrome diagonal. Now, returning the removed squares, we obtain a rectangle $(N-1) \times N$, and our aim is to supplement it on the top by one more flag.

If Q has a yellow diagonal, then we can take each flag with a yellow first square (it exists by a choice of the first column; moreover, it is not used in Q). Conversely, if the diagonal of Q is blue then we can take any of the $\geq 2^{N-3} + 1 - (N-1) > 0$ remaining good flags. So, in both cases we get a desired $N \times N$ square.

Solution 2. We present a different proof of the estimate $M_N \leq 2^{N-2} + 1$. We do not use the induction, involving Hall's lemma on matchings instead.

Consider arbitrary $2^{N-2} + 1$ distinct flags and arrange them into a large $(2^{N-2} + 1) \times N$ flag. Construct two bipartite graphs $G_{\mathrm{y}} = (V \cup V', E_{\mathrm{y}})$ and $G_{\mathrm{b}} = (V \cup V', E_{\mathrm{b}})$ with the common set of vertices as follows. Let V and V' be the set of columns and the set of flags under consideration, respectively. Next, let the edge (c, f) appear in E_{y} if the intersection of column c and flag f is yellow, and $(c, f) \in E_{\mathrm{b}}$ otherwise. Then we have to prove exactly that one of the graphs G_{y} and G_{b} contains a matching with all the vertices of V involved.

Assume that these matchings do not exist. By Hall's lemma, it means that there exist two sets of columns $S_{\mathrm{y}}, S_{\mathrm{b}} \subset V$ such that $|E_{\mathrm{y}}(S_{\mathrm{y}})| \leq |S_{\mathrm{y}}| - 1$ and $|E_{\mathrm{b}}(S_{\mathrm{b}})| \leq |S_{\mathrm{b}}| - 1$ (in the left-hand sides, $E_{\mathrm{y}}(S_{\mathrm{y}})$ and $E_{\mathrm{b}}(S_{\mathrm{b}})$ denote respectively the sets of all vertices connected to S_{y} and S_{b} in the corresponding graphs). Our aim is to prove that this is impossible. Note that $S_{\mathrm{y}}, S_{\mathrm{b}} \neq V$ since $N \leq 2^{N-2} + 1$.

First, suppose that $S_{\mathrm{y}} \cap S_{\mathrm{b}} \neq \varnothing$, so there exists some $c \in S_{\mathrm{y}} \cap S_{\mathrm{b}}$. Note that each flag is connected to c either in G_{y} or in G_{b}, hence $E_{\mathrm{y}}(S_{\mathrm{y}}) \cup E_{\mathrm{b}}(S_{\mathrm{b}}) = V'$. Hence we have $2^{N-2} + 1 = |V'| \leq |E_{\mathrm{y}}(S_{\mathrm{y}})| + |E_{\mathrm{b}}(S_{\mathrm{b}})| \leq |S_{\mathrm{y}}| + |S_{\mathrm{b}}| - 2 \leq 2N - 4$; this is impossible for $N \geq 4$.

So, we have $S_{\mathrm{y}} \cap S_{\mathrm{b}} = \varnothing$. Let $y = |S_{\mathrm{y}}|$, $b = |S_{\mathrm{b}}|$. From the construction of our graph, we have that all the flags in the set $V'' = V' \setminus \big(E_{\mathrm{y}}(S_{\mathrm{y}}) \cup E_{\mathrm{b}}(S_{\mathrm{b}})\big)$ have blue squares in the columns of S_{y} and yellow squares in the columns of S_{b}. Hence the only undetermined positions in these flags are the remaining $N - y - b$ ones, so $2^{N-y-b} \geq |V''| \geq |V'| - |E_{\mathrm{y}}(S_{\mathrm{y}})| - |E_{\mathrm{b}}(S_{\mathrm{b}})| \geq 2^{N-2} + 1 - (y-1) - (b-1)$, or, denoting $c = y + b$, $2^{N-c} + c > 2^{N-2} + 2$. This is impossible since $N \geq c \geq 2$.

C3. 2500 chess kings have to be placed on a 100×100 chessboard so that

(i) no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);

(ii) each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

(Russia)

Answer. There are two such arrangements.

Solution. Suppose that we have an arrangement satisfying the problem conditions. Divide the board into 2×2 pieces; we call these pieces *blocks*. Each block can contain not more than one king (otherwise these two kings would attack each other); hence, by the pigeonhole principle each block must contain exactly one king.

Now assign to each block a letter T or B if a king is placed in its top or bottom half, respectively. Similarly, assign to each block a letter L or R if a king stands in its left or right half. So we define *T-blocks*, *B-blocks*, *L-blocks*, and *R-blocks*. We also combine the letters; we call a block *a TL-block* if it is simultaneously T-block and L-block. Similarly we define *TR-blocks*, *BL-blocks*, and *BR-blocks*. The arrangement of blocks determines uniquely the arrangement of kings; so in the rest of the solution we consider the 50×50 system of blocks (see Fig. 1). We identify the blocks by their coordinate pairs; the pair (i, j), where $1 \le i, j \le 50$, refers to the jth block in the ith row (or the ith block in the jth column). The upper-left block is $(1, 1)$.

The system of blocks has the following properties..

(i$'$) If (i, j) is a B-block then $(i+1, j)$ is a B-block: otherwise the kings in these two blocks can take each other. Similarly: if (i, j) is a T-block then $(i-1, j)$ is a T-block; if (i, j) is an L-block then $(i, j-1)$ is an L-block; if (i, j) is an R-block then $(i, j+1)$ is an R-block.

(ii$'$) Each column contains exactly 25 L-blocks and 25 R-blocks, and each row contains exactly 25 T-blocks and 25 B-blocks. In particular, the total number of L-blocks (or R-blocks, or T-blocks, or B-blocks) is equal to $25 \cdot 50 = 1250$.

Consider any B-block of the form $(1, j)$. By (i$'$), all blocks in the jth column are B-blocks; so we call such a column *B-column*. By (ii$'$), we have 25 B-blocks in the first row, so we obtain 25 B-columns. These 25 B-columns contain 1250 B-blocks, hence all blocks in the remaining columns are T-blocks, and we obtain 25 *T-columns*. Similarly, there are exactly 25 *L-rows* and exactly 25 *R-rows*.

Now consider an arbitrary pair of a T-column and a neighboring B-column (columns with numbers j and $j+1$).

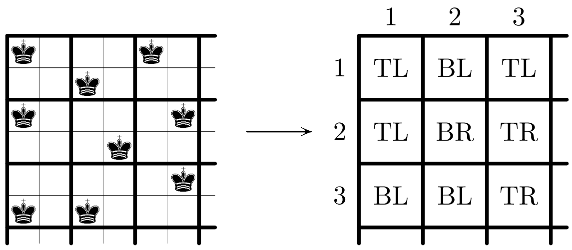


Fig. 1

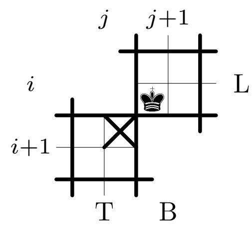


Fig. 2

Case 1. Suppose that the jth column is a T-column, and the $(j+1)$th column is a B-column. Consider some index i such that the ith row is an L-row; then $(i, j+1)$ is a BL-block. Therefore, $(i+1, j)$ cannot be a TR-block (see Fig. 2), hence $(i+1, j)$ is a TL-block, thus the

$(i+1)$th row is an L-row. Now, choosing the ith row to be the topmost L-row, we successively obtain that all rows from the ith to the 50th are L-rows. Since we have exactly 25 L-rows, it follows that the rows from the 1st to the 25th are R-rows, and the rows from the 26th to the 50th are L-rows.

Now consider the neighboring R-row and L-row (that are the rows with numbers 25 and 26). Replacing in the previous reasoning rows by columns and vice versa, the columns from the 1st to the 25th are T-columns, and the columns from the 26th to the 50th are B-columns. So we have a unique arrangement of blocks that leads to the arrangement of kings satisfying the condition of the problem (see Fig. 3).

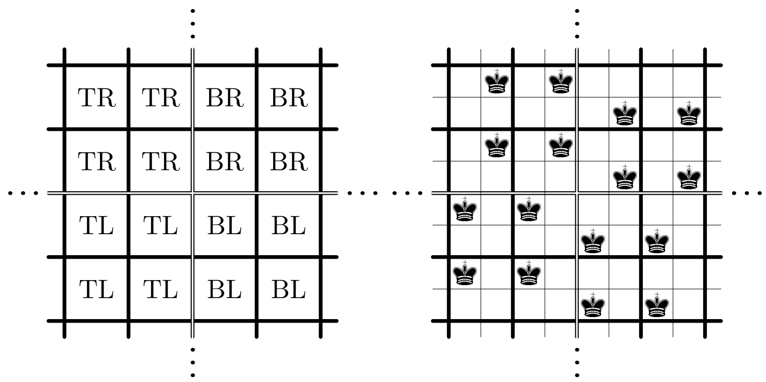


Fig. 3

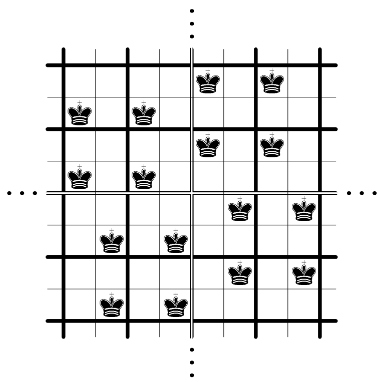


Fig. 4

Case 2. Suppose that the jth column is a B-column, and the $(j+1)$th column is a T-column. Repeating the arguments from Case 1, we obtain that the rows from the 1st to the 25th are L-rows (and all other rows are R-rows), the columns from the 1st to the 25th are B-columns (and all other columns are T-columns), so we find exactly one more arrangement of kings (see Fig. 4).

C4. Six stacks $S_1, \dots, S_6$ of coins are standing in a row. In the beginning every stack contains a single coin. There are two types of allowed moves:

Move 1: If stack S_k with $1 \le k \le 5$ contains at least one coin, you may remove one coin from S_k and add two coins to S_{k+1}.

Move 2: If stack S_k with $1 \le k \le 4$ contains at least one coin, then you may remove one coin from S_k and exchange stacks S_{k+1} and S_{k+2}.

Decide whether it is possible to achieve by a sequence of such moves that the first five stacks are empty, whereas the sixth stack S_6 contains exactly $2010^{2010^{2010}}$ coins.

C4′. Same as Problem C4, but the constant $2010^{2010^{2010}}$ is replaced by 2010^{2010}.

(Netherlands)

Answer. Yes (in both variants of the problem). There exists such a sequence of moves.

Solution. Denote by $(a_1, a_2, \dots, a_n) \to (a'_1, a'_2, \dots, a'_n)$ the following: if some consecutive stacks contain $a_1, \dots, a_n$ coins, then it is possible to perform several allowed moves such that the stacks contain $a'_1, \dots, a'_n$ coins respectively, whereas the contents of the other stacks remain unchanged.

Let $A = 2010^{2010}$ or $A = 2010^{2010^{2010}}$, respectively. Our goal is to show that

$$(1,1,1,1,1,1) \to (0,0,0,0,0,A).$$

First we prove two auxiliary observations.

Lemma 1. $(a, 0, 0) \to (0, 2^a, 0)$ for every $a \ge 1$.

Proof. We prove by induction that $(a, 0, 0) \to (a-k, 2^k, 0)$ for every $1 \le k \le a$. For $k = 1$, apply Move 1 to the first stack:

$$(a, 0, 0) \to (a-1, 2, 0) = (a-1, 2^1, 0).$$

Now assume that $k < a$ and the statement holds for some $k < a$. Starting from $(a-k, 2^k, 0)$, apply Move 1 to the middle stack 2^k times, until it becomes empty. Then apply Move 2 to the first stack:

$$(a-k, 2^k, 0) \to (a-k, 2^k-1, 2) \to \dots \to (a-k, 0, 2^{k+1}) \to (a-k-1, 2^{k+1}, 0).$$

Hence,

$$(a, 0, 0) \to (a-k, 2^k, 0) \to (a-k-1, 2^{k+1}, 0). \qquad \square$$

Lemma 2. For every positive integer n, let $P_n = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n}$ (e.g. $P_3 = 2^{2^2} = 16$). Then $(a, 0, 0, 0) \to (0, P_a, 0, 0)$ for every $a \ge 1$.

Proof. Similarly to Lemma 1, we prove that $(a, 0, 0, 0) \to (a-k, P_k, 0, 0)$ for every $1 \le k \le a$.

For $k = 1$, apply Move 1 to the first stack:

$$(a, 0, 0, 0) \to (a-1, 2, 0, 0) = (a-1, P_1, 0, 0).$$

Now assume that the lemma holds for some $k < a$. Starting from $(a-k, P_k, 0, 0)$, apply Lemma 1, then apply Move 1 to the first stack:

$$(a-k, P_k, 0, 0) \to (a-k, 0, 2^{P_k}, 0) = (a-k, 0, P_{k+1}, 0) \to (a-k-1, P_{k+1}, 0, 0).$$

Therefore,

$$(a, 0, 0, 0) \to (a-k, P_k, 0, 0) \to (a-k-1, P_{k+1}, 0, 0). \qquad \square$$

Now we prove the statement of the problem.

First apply Move 1 to stack 5, then apply Move 2 to stacks S_4, S_3, S_2 and S_1 in this order. Then apply Lemma 2 twice:

$$(1,1,1,1,1,1) \to (1,1,1,1,0,3) \to (1,1,1,0,3,0) \to (1,1,0,3,0,0) \to (1,0,3,0,0,0) \to \\ \to (0,3,0,0,0,0) \to (0,0,P_3,0,0,0) = (0,0,16,0,0,0) \to (0,0,0,P_{16},0,0).$$

We already have more than A coins in stack S_4, since

$$A \le 2010^{2010^{2010}} < (2^{11})^{2010^{2010}} = 2^{11\cdot 2010^{2010}} < 2^{2010^{2011}} < 2^{(2^{11})^{2011}} = 2^{2^{11\cdot 2011}} < 2^{2^{2^{15}}} < P_{16}.$$

To decrease the number of coins in stack S_4, apply Move 2 to this stack repeatedly until its size decreases to $A/4$. (In every step, we remove a coin from S_4 and exchange the empty stacks S_5 and S_6.)

$$(0,0,0,P_{16},0,0) \to (0,0,0,P_{16}-1,0,0) \to (0,0,0,P_{16}-2,0,0) \to \\ \to \cdots \to (0,0,0,A/4,0,0).$$

Finally, apply Move 1 repeatedly to empty stacks S_4 and S_5:

$$(0,0,0,A/4,0,0) \to \cdots \to (0,0,0,0,A/2,0) \to \cdots \to (0,0,0,0,0,A).$$

Comment 1. Starting with only 4 stack, it is not hard to check manually that we can achieve at most 28 coins in the last position. However, around 5 and 6 stacks the maximal number of coins explodes. With 5 stacks it is possible to achieve more than $2^{2^{14}}$ coins. With 6 stacks the maximum is greater than $P_{P_{2^{14}}}$.

It is not hard to show that the numbers 2010^{2010} and $2010^{2010^{2010}}$ in the problem can be replaced by any nonnegative integer up to $P_{P_{2^{14}}}$.

Comment 2. The simpler variant C4′ of the problem can be solved without Lemma 2:

$$(1,1,1,1,1,1) \to (0,3,1,1,1,1) \to (0,1,5,1,1,1) \to (0,1,1,9,1,1) \to \\ \to (0,1,1,1,17,1) \to (0,1,1,1,0,35) \to (0,1,1,0,35,0) \to (0,1,0,35,0,0) \to \\ \to (0,0,35,0,0,0) \to (0,0,1,2^{34},0,0) \to (0,0,1,0,2^{2^{34}},0) \to (0,0,0,2^{2^{34}},0,0) \\ \to (0,0,0,2^{2^{34}}-1,0,0) \to \ldots \to (0,0,0,A/4,0,0) \to (0,0,0,0,A/2,0) \to (0,0,0,0,0,A).$$

For this reason, the PSC suggests to consider the problem C4 as well. Problem C4 requires more invention and technical care. On the other hand, the problem statement in C4′ hides the fact that the resulting amount of coins can be such incredibly huge and leaves this discovery to the students.

C5. $n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let w_i and ℓ_i be respectively the number of wins and losses of the ith player. Prove that

$$\sum_{i=1}^{n}(w_i - \ell_i)^3 \geq 0. \tag{1}$$

(South Korea)

Solution. For any tournament T satisfying the problem condition, denote by $S(T)$ sum under consideration, namely

$$S(T) = \sum_{i=1}^{n}(w_i - \ell_i)^3.$$

First, we show that the statement holds if a tournament T has only 4 players. Actually, let $A = (a_1, a_2, a_3, a_4)$ be the number of wins of the players; we may assume that $a_1 \geq a_2 \geq a_3 \geq a_4$. We have $a_1 + a_2 + a_3 + a_4 = \binom{4}{2} = 6$, hence $a_4 \leq 1$. If $a_4 = 0$, then we cannot have $a_1 = a_2 = a_3 = 2$, otherwise the company of all players is bad. Hence we should have $A = (3, 2, 1, 0)$, and $S(T) = 3^3 + 1^3 + (-1)^3 + (-3)^3 = 0$. On the other hand, if $a_4 = 1$, then only two possibilities, $A = (3, 1, 1, 1)$ and $A = (2, 2, 1, 1)$ can take place. In the former case we have $S(T) = 3^3 + 3 \cdot (-2)^3 > 0$, while in the latter one $S(T) = 1^3 + 1^3 + (-1)^3 + (-1)^3 = 0$, as desired.

Now we turn to the general problem. Consider a tournament T with no bad companies and enumerate the players by the numbers from 1 to n. For every 4 players i_1, i_2, i_3, i_4 consider a "sub-tournament" $T_{i_1 i_2 i_3 i_4}$ consisting of only these players and the games which they performed with each other. By the abovementioned, we have $S(T_{i_1 i_2 i_3 i_4}) \geq 0$. Our aim is to prove that

$$S(T) = \sum_{i_1, i_2, i_3, i_4} S(T_{i_1 i_2 i_3 i_4}), \tag{2}$$

where the sum is taken over all 4-tuples of distinct numbers from the set $\{1, \ldots, n\}$. This way the problem statement will be established.

We interpret the number $(w_i - \ell_i)^3$ as following. For $i \neq j$, let $\varepsilon_{ij} = 1$ if the ith player wins against the jth one, and $\varepsilon_{ij} = -1$ otherwise. Then

$$(w_i - \ell_i)^3 = \left(\sum_{j \neq i} \varepsilon_{ij}\right)^3 = \sum_{j_1, j_2, j_3 \neq i} \varepsilon_{ij_1} \varepsilon_{ij_2} \varepsilon_{ij_3}.$$

Hence,

$$S(T) = \sum_{i \notin \{j_1, j_2, j_3\}} \varepsilon_{ij_1} \varepsilon_{ij_2} \varepsilon_{ij_3}.$$

To simplify this expression, consider all the terms in this sum where two indices are equal. If, for instance, $j_1 = j_2$, then the term contains $\varepsilon_{ij_1}^2 = 1$, so we can replace this term by ε_{ij_3}. Make such replacements for each such term; obviously, after this change each term of the form ε_{ij_3} will appear $P(T)$ times, hence

$$S(T) = \sum_{|\{i, j_1, j_2, j_3\}| = 4} \varepsilon_{ij_1} \varepsilon_{ij_2} \varepsilon_{ij_3} + P(T) \sum_{i \neq j} \varepsilon_{ij} = S_1(T) + P(T) S_2(T).$$

We show that $S_2(T) = 0$ and hence $S(T) = S_1(T)$ for each tournament. Actually, note that $\varepsilon_{ij} = -\varepsilon_{ji}$, and the whole sum can be split into such pairs. Since the sum in each pair is 0, so is $S_2(T)$.

Thus the desired equality (2) rewrites as

$$S_1(T) = \sum_{i_1,i_2,i_3,i_4} S_1(T_{i_1 i_2 i_3 i_4}). \tag{3}$$

Now, if all the numbers j_1, j_2, j_3 are distinct, then the set $\{i, j_1, j_2, j_3\}$ is contained in exactly one 4-tuple, hence the term $\varepsilon_{ij_1}\varepsilon_{ij_2}\varepsilon_{ij_3}$ appears in the right-hand part of (3) exactly once, as well as in the left-hand part. Clearly, there are no other terms in both parts, so the equality is established.

Solution 2. Similarly to the first solution, we call the subsets of players as *companies*, and the k-element subsets will be called as *k-companies*.

In any company of the players, call a player *the local champion of the company* if he defeated all other members of the company. Similarly, if a player lost all his games against the others in the company then call him *the local loser of the company*. Obviously every company has at most one local champion and at most one local loser. By the condition of the problem, whenever a 4-company has a local loser, then this company has a local champion as well.

Suppose that k is some positive integer, and let us count all cases when a player is the local champion of some k-company. The ith player won against w_i other player. To be the local champion of a k-company, he must be a member of the company, and the other $k-1$ members must be chosen from those whom he defeated. Therefore, the ith player is the local champion of $\binom{w_i}{k-1}$ k-companies. Hence, the total number of local champions of all k-companies is $\sum_{i=1}^{n}\binom{w_i}{k-1}$.

Similarly, the total number of local losers of the k-companies is $\sum_{i=1}^{n}\binom{\ell_i}{k-1}$.

Now apply this for $k = 2, 3$ and 4.

Since every game has a winner and a loser, we have $\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} \ell_i = \binom{n}{2}$, and hence

$$\sum_{i=1}^{n}\big(w_i - \ell_i\big) = 0. \tag{4}$$

In every 3-company, either the players defeated one another in a cycle or the company has both a local champion and a local loser. Therefore, the total number of local champions and local losers in the 3-companies is the same, $\sum_{i=1}^{n}\binom{w_i}{2} = \sum_{i=1}^{n}\binom{\ell_i}{2}$. So we have

$$\sum_{i=1}^{n}\left(\binom{w_i}{2} - \binom{\ell_i}{2}\right) = 0. \tag{5}$$

In every 4-company, by the problem's condition, the number of local losers is less than or equal to the number of local champions. Then the same holds for the total numbers of local champions and local losers in all 4-companies, so $\sum_{i=1}^{n}\binom{w_i}{3} \geq \sum_{i=1}^{n}\binom{\ell_i}{3}$. Hence,

$$\sum_{i=1}^{n}\left(\binom{w_i}{3}-\binom{\ell_i}{3}\right) \geq 0. \tag{6}$$

Now we establish the problem statement (1) as a linear combination of (4), (5) and (6). It is easy check that

$$(x-y)^3 = 24\left(\binom{x}{3}-\binom{y}{3}\right) + 24\left(\binom{x}{2}-\binom{y}{2}\right) - \big(3(x+y)^2-4\big)(x-y).$$

Apply this identity to $x = w_1$ and $y = \ell_i$. Since every player played $n-1$ games, we have $w_i + \ell_i = n-1$, and thus

$$(w_i-\ell_i)^3 = 24\left(\binom{w_i}{3}-\binom{\ell_i}{3}\right) + 24\left(\binom{w_i}{2}-\binom{\ell_i}{2}\right) - \big(3(n-1)^2-4\big)\Big(w_i-\ell_i\Big).$$

Then

$$\sum_{i=1}^{n}(w_i-\ell_i)^3 = 24\underbrace{\sum_{i=1}^{n}\left(\binom{w_i}{3}-\binom{\ell_i}{3}\right)}_{\geq 0} + 24\underbrace{\sum_{i=1}^{n}\left(\binom{w_i}{2}-\binom{\ell_i}{2}\right)}_{0} - \big(3(n-1)^2-4\big)\underbrace{\sum_{i=1}^{n}\Big(w_i-\ell_i\Big)}_{0} \geq 0.$$

C6. Given a positive integer k and other two integers $b > w > 1$. There are two strings of pearls, a string of b black pearls and a string of w white pearls. The *length* of a string is the number of pearls on it.

One cuts these strings in some steps by the following rules. In each step:

(i) The strings are ordered by their lengths in a non-increasing order. If there are some strings of equal lengths, then the white ones precede the black ones. Then k first ones (if they consist of more than one pearl) are chosen; if there are less than k strings longer than 1, then one chooses all of them.

(ii) Next, one cuts each chosen string into two parts differing in length by at most one.

(For instance, if there are strings of 5, 4, 4, 2 black pearls, strings of 8, 4, 3 white pearls and $k = 4$, then the strings of 8 white, 5 black, 4 white and 4 black pearls are cut into the parts $(4, 4)$, $(3, 2)$, $(2, 2)$ and $(2, 2)$, respectively.)

The process stops immediately after the step when a first isolated white pearl appears. Prove that at this stage, there will still exist a string of at least two black pearls.

(Canada)

Solution 1. Denote the situation after the ith step by A_i; hence A_0 is the initial situation, and $A_{i-1} \to A_i$ is the ith step. We call a string containing m pearls an m-string; it is an *m-w-string* or a *m-b-string* if it is white or black, respectively.

We continue the process until every string consists of a single pearl. We will focus on three moments of the process: (a) the first stage A_s when the first 1-string (no matter black or white) appears; (b) the first stage A_t where the total number of strings is greater than k (if such moment does not appear then we put $t = \infty$); and (c) the first stage A_f when all black pearls are isolated. It is sufficient to prove that in A_{f-1} (or earlier), a 1-w-string appears.

We start with some easy properties of the situations under consideration. Obviously, we have $s \leq f$. Moreover, all b-strings from A_{f-1} become single pearls in the fth step, hence all of them are 1- or 2-b-strings.

Next, observe that in each step $A_i \to A_{i+1}$ with $i \leq t - 1$, all (>1)-strings were cut since there are not more than k strings at all; if, in addition, $i < s$, then there were no 1-string, so all the strings were cut in this step.

Now, let B_i and b_i be the lengths of the longest and the shortest b-strings in A_i, and let W_i and w_i be the same for w-strings. We show by induction on $i \leq \min\{s, t\}$ that (i) the situation A_i contains exactly 2^i black and 2^i white strings, (ii) $B_i \geq W_i$, and (iii) $b_i \geq w_i$. The base case $i = 0$ is obvious. For the induction step, if $i \leq \min\{s, t\}$ then in the ith step, each string is cut, thus the claim (i) follows from the induction hypothesis; next, we have $B_i = \lceil B_{i-1}/2 \rceil \geq \lceil W_{i-1}/2 \rceil = W_i$ and $b_i = \lfloor b_{i-1}/2 \rfloor \geq \lfloor w_{i-1}/2 \rfloor = w_i$, thus establishing (ii) and (iii).

For the numbers s, t, f, two cases are possible.

Case 1. Suppose that $s \leq t$ or $f \leq t + 1$ (and hence $s \leq t + 1$); in particular, this is true when $t = \infty$. Then in A_{s-1} we have $B_{s-1} \geq W_{s-1}$, $b_{s-1} \geq w_{s-1} > 1$ as $s - 1 \leq \min\{s, t\}$. Now, if $s = f$, then in A_{s-1}, there is no 1-w-string as well as no (>2)-b-string. That is, $2 = B_{s-1} \geq W_{s-1} \geq b_{s-1} \geq w_{s-1} > 1$, hence all these numbers equal 2. This means that in A_{s-1}, all strings contain 2 pearls, and there are 2^{s-1} black and 2^{s-1} white strings, which means $b = 2 \cdot 2^{s-1} = w$. This contradicts the problem conditions.

Hence we have $s \leq f - 1$ and thus $s \leq t$. Therefore, in the sth step each string is cut into two parts. Now, if a 1-b-string appears in this step, then from $w_{s-1} \leq b_{s-1}$ we see that a

1-w-string appears as well; so, in each case in the sth step a 1-w-string appears, while not all black pearls become single, as desired.

Case 2. Now assume that $t+1 \le s$ and $t+2 \le f$. Then in A_t we have exactly 2^t white and 2^t black strings, all being larger than 1, and $2^{t+1} > k \ge 2^t$ (the latter holds since 2^t is the total number of strings in A_{t-1}). Now, in the $(t+1)$st step, exactly k strings are cut, not more than 2^t of them being black; so the number of w-strings in A_{t+1} is at least $2^t + (k - 2^t) = k$. Since the number of w-strings does not decrease in our process, in A_{f-1} we have at least k white strings as well.

Finally, in A_{f-1}, all b-strings are not larger than 2, and at least one 2-b-string is cut in the fth step. Therefore, at most $k-1$ white strings are cut in this step, hence there exists a w-string $\mathcal{W}$ which is not cut in the fth step. On the other hand, since a 2-b-string is cut, all ($\ge$2)-w-strings should also be cut in the fth step; hence $\mathcal{W}$ should be a single pearl. This is exactly what we needed.

Comment. In this solution, we used the condition $b \ne w$ only to avoid the case $b = w = 2^t$. Hence, if a number $b = w$ is not a power of 2, then the problem statement is also valid.

Solution 2. We use the same notations as introduced in the first paragraph of the previous solution. We claim that at every stage, there exist a u-b-string and a v-w-string such that either

(i) $u > v \ge 1$, or

(ii) $2 \le u \le v < 2u$, and there also exist $k-1$ of ($>v/2$)-strings other than considered above.

First, we notice that this statement implies the problem statement. Actually, in both cases (i) and (ii) we have $u > 1$, so at each stage there exists a ($\ge$2)-b-string, and for the last stage it is exactly what we need.

Now, we prove the claim by induction on the number of the stage. Obviously, for A_0 the condition (i) holds since $b > w$. Further, we suppose that the statement holds for A_i, and prove it for A_{i+1}. Two cases are possible.

Case 1. Assume that in A_i, there are a u-b-string and a v-w-string with $u > v$. We can assume that v is the length of the shortest w-string in A_i; since we are not at the final stage, we have $v \ge 2$. Now, in the $(i+1)$st step, two subcases may occur.

Subcase 1a. Suppose that either no u-b-string is cut, or both some u-b-string and some v-w-string are cut. Then in A_{i+1}, we have either a u-b-string and a ($\le v$)-w-string (and (i) is valid), or we have a $\lceil u/2 \rceil$-b-string and a $\lfloor v/2 \rfloor$-w-string. In the latter case, from $u > v$ we get $\lceil u/2 \rceil > \lfloor v/2 \rfloor$, and (i) is valid again.

Subcase 1b. Now, some u-b-string is cut, and no v-w-string is cut (and hence all the strings which are cut are longer than v). If $u' = \lceil u/2 \rceil > v$, then the condition (i) is satisfied since we have a u'-b-string and a v-w-string in A_{i+1}. Otherwise, notice that the inequality $u > v \ge 2$ implies $u' \ge 2$. Furthermore, besides a fixed u-b-string, other $k-1$ of ($\ge v+1$)-strings should be cut in the $(i+1)$st step, hence providing at least $k-1$ of ($\ge \lceil (v+1)/2 \rceil$)-strings, and $\lceil (v+1)/2 \rceil > v/2$. So, we can put $v' = v$, and we have $u' \le v < u \le 2u'$, so the condition (ii) holds for A_{i+1}.

Case 2. Conversely, assume that in A_i there exist a u-b-string, a v-w-string ($2 \le u \le v < 2u$) and a set S of $k-1$ other strings larger than $v/2$ (and hence larger than 1). In the $(i+1)$st step, three subcases may occur.

Subcase 2a. Suppose that some u-b-string is not cut, and some v-w-string is cut. The latter one results in a $\lfloor v/2 \rfloor$-w-string, we have $v' = \lfloor v/2 \rfloor < u$, and the condition (i) is valid.

Subcase 2b. Next, suppose that no v-w-string is cut (and therefore no u-b-string is cut as $u \leq v$). Then all k strings which are cut have the length $> v$, so each one results in a $(>v/2)$-string. Hence in A_{i+1}, there exist $k \geq k-1$ of $(>v/2)$-strings other than the considered u- and v-strings, and the condition (ii) is satisfied.

Subcase 2c. In the remaining case, all u-b-strings are cut. This means that all $(\geq u)$-strings are cut as well, hence our v-w-string is cut. Therefore in A_{i+1} there exists a $\lceil u/2 \rceil$-b-string together with a $\lfloor v/2 \rfloor$-w-string. Now, if $u' = \lceil u/2 \rceil > \lfloor v/2 \rfloor = v'$ then the condition (i) is fulfilled. Otherwise, we have $u' \leq v' < u \leq 2u'$. In this case, we show that $u' \geq 2$. If, to the contrary, $u' = 1$ (and hence $u = 2$), then all black and white (≥ 2)-strings should be cut in the $(i+1)$st step, and among these strings there are at least a u-b-string, a v-w-string, and $k-1$ strings in S ($k+1$ strings altogether). This is impossible.

Hence, we get $2 \leq u' \leq v' < 2u'$. To reach (ii), it remains to check that in A_{i+1}, there exists a set S' of $k-1$ other strings larger than $v'/2$. These will be exactly the strings obtained from the elements of S. Namely, each $s \in S$ was either cut in the $(i+1)$st step, or not. In the former case, let us include into S' the largest of the strings obtained from s; otherwise we include s itself into S'. All $k-1$ strings in S' are greater than $v/2 \geq v'$, as desired.

C7. Let $P_1, \dots, P_s$ be arithmetic progressions of integers, the following conditions being satisfied:

(i) each integer belongs to at least one of them;

(ii) each progression contains a number which does not belong to other progressions.

Denote by n the least common multiple of steps of these progressions; let $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be its prime factorization. Prove that

$$s \geq 1 + \sum_{i=1}^{k} \alpha_i (p_i - 1).$$

(Germany)

Solution 1. First, we prove the key lemma, and then we show how to apply it to finish the solution.

Let $n_1, \dots, n_k$ be positive integers. By an $n_1 \times n_2 \times \cdots \times n_k$ *grid* we mean the set $N = \{(a_1, \dots, a_k) : a_i \in \mathbb{Z},\ 0 \leq a_i \leq n_i - 1\}$; the elements of N will be referred to as *points*. In this grid, we define *a subgrid* as a subset of the form

$$L = \{(b_1, \dots, b_k) \in N : b_{i_1} = x_{i_1}, \dots, b_{i_t} = x_{i_t}\}, \tag{1}$$

where $I = \{i_1, \dots, i_t\}$ is an arbitrary nonempty set of indices, and $x_{i_j} \in [0, n_{i_j} - 1]$ $(1 \leq j \leq t)$ are fixed integer numbers. Further, we say that a subgrid (1) is *orthogonal* to the ith coordinate axis if $i \in I$, and that it is *parallel* to the ith coordinate axis otherwise.

Lemma. Assume that the grid N is covered by subgrids $L_1, L_2, \dots, L_s$ (this means $N = \bigcup_{i=1}^{s} L_i$) so that

(ii′) each subgrid contains a point which is not covered by other subgrids;

(iii) for each coordinate axis, there exists a subgrid L_i orthogonal to this axis.

Then

$$s \geq 1 + \sum_{i=1}^{k} (n_i - 1).$$

Proof. Assume to the contrary that $s \leq \sum_i (n_i - 1) = s'$. Our aim is to find a point that is not covered by $L_1, \dots, L_s$.

The idea of the proof is the following. Imagine that we expand each subgrid to some maximal subgrid so that for the ith axis, there will be at most $n_i - 1$ maximal subgrids orthogonal to this axis. Then the desired point can be found easily: its ith coordinate should be that not covered by the maximal subgrids orthogonal to the ith axis. Surely, the conditions for existence of such expansion are provided by Hall's lemma on matchings. So, we will follow this direction, although we will apply Hall's lemma to some subgraph instead of the whole graph.

Construct a bipartite graph $G = (V \cup V', E)$ as follows. Let $V = \{L_1, \dots, L_s\}$, and let $V' = \{v_{ij} : 1 \leq i \leq s,\ 1 \leq j \leq n_i - 1\}$ be some set of s' elements. Further, let the edge (L_m, v_{ij}) appear iff L_m is orthogonal to the ith axis.

For each subset $W \subset V$, denote

$$f(W) = \{v \in V' : (L, v) \in E \text{ for some } L \in W\}.$$

Notice that $f(V) = V'$ by (iii).

Now, consider the set $W \subset V$ containing the maximal number of elements such that $|W| > |f(W)|$; if there is no such set then we set $W = \varnothing$. Denote $W' = f(W)$, $U = V \backslash W$, $U' = V' \backslash W'$.

By our assumption and the Lemma condition, $|f(V)| = |V'| \geq |V|$, hence $W \neq V$ and $U \neq \varnothing$. Permuting the coordinates, we can assume that $U' = \{v_{ij} : 1 \leq i \leq \ell\}$, $W' = \{v_{ij} : \ell{+}1 \leq i \leq k\}$.

Consider the induced subgraph G' of G on the vertices $U \cup U'$. We claim that for every $X \subset U$, we get $|f(X) \cap U'| \geq |X|$ (so G' satisfies the conditions of Hall's lemma). Actually, we have $|W| \geq |f(W)|$, so if $|X| > |f(X) \cap U'|$ for some $X \subset U$, then we have

$$|W \cup X| = |W| + |X| > |f(W)| + |f(X) \cap U'| = |f(W) \cup (f(X) \cap U')| = |f(W \cup X)|.$$

This contradicts the maximality of $|W|$.

Thus, applying Hall's lemma, we can assign to each $L \in U$ some vertex $v_{ij} \in U'$ so that to distinct elements of U, distinct vertices of U' are assigned. In this situation, we say that $L \in U$ corresponds to the ith axis, and write $g(L) = i$. Since there are $n_i - 1$ vertices of the form v_{ij}, we get that for each $1 \leq i \leq \ell$, not more than $n_i - 1$ subgrids correspond to the ith axis.

Finally, we are ready to present the desired point. Since $W \neq V$, there exists a point $b = (b_1, b_2, \ldots, b_k) \in N \backslash (\cup_{L \in W} L)$. On the other hand, for every $1 \leq i \leq \ell$, consider any subgrid $L \in U$ with $g(L) = i$. This means exactly that L is orthogonal to the ith axis, and hence all its elements have the same ith coordinate c_L. Since there are at most $n_i - 1$ such subgrids, there exists a number $0 \leq a_i \leq n_i - 1$ which is not contained in a set $\{c_L : g(L) = i\}$. Choose such number for every $1 \leq i \leq \ell$. Now we claim that point $a = (a_1, \ldots, a_\ell, b_{\ell+1}, \ldots, b_k)$ is not covered, hence contradicting the Lemma condition.

Surely, point a cannot lie in some $L \in U$, since all the points in L have $g(L)$th coordinate $c_L \neq a_{g(L)}$. On the other hand, suppose that $a \in L$ for some $L \in W$; recall that $b \notin L$. But the points a and b differ only at first ℓ coordinates, so L should be orthogonal to at least one of the first ℓ axes, and hence our graph contains some edge (L, v_{ij}) for $i \leq \ell$. It contradicts the definition of W'. The Lemma is proved. □

Now we turn to the problem. Let d_j be the step of the progression P_j. Note that since $n = \text{l.c.m.}(d_1, \ldots, d_s)$, for each $1 \leq i \leq k$ there exists an index $j(i)$ such that $p_i^{\alpha_i} \mid d_{j(i)}$. We assume that $n > 1$; otherwise the problem statement is trivial.

For each $0 \leq m \leq n-1$ and $1 \leq i \leq k$, let m_i be the residue of m modulo $p_i^{\alpha_i}$, and let $m_i = \overline{r_{i\alpha_i} \ldots r_{i1}}$ be the base p_i representation of m_i (possibly, with some leading zeroes). Now, we put into correspondence to m the sequence $r(m) = (r_{11}, \ldots, r_{1\alpha_1}, r_{21}, \ldots, r_{k\alpha_k})$. Hence $r(m)$ lies in a $\underbrace{p_1 \times \cdots \times p_1}_{\alpha_1 \text{ times}} \times \cdots \times \underbrace{p_k \times \cdots \times p_k}_{\alpha_k \text{ times}}$ grid N.

Surely, if $r(m) = r(m')$ then $p_i^{\alpha_i} \mid m_i - m_i'$, which follows $p_i^{\alpha_i} \mid m - m'$ for all $1 \leq i \leq k$; consequently, $n \mid m - m'$. So, when m runs over the set $\{0, \ldots, n-1\}$, the sequences $r(m)$ do not repeat; since $|N| = n$, this means that r is a bijection between $\{0, \ldots, n-1\}$ and N. Now we will show that for each $1 \leq i \leq s$, the set $L_i = \{r(m) : m \in P_i\}$ is a subgrid, and that for each axis there exists a subgrid orthogonal to this axis. Obviously, these subgrids cover N, and the condition (ii′) follows directly from (ii). Hence the Lemma provides exactly the estimate we need.

Consider some $1 \leq j \leq s$ and let $d_j = p_1^{\gamma_1} \ldots p_k^{\gamma_k}$. Consider some $q \in P_j$ and let $r(q) = (r_{11}, \ldots, r_{k\alpha_k})$. Then for an arbitrary q', setting $r(q') = (r'_{11}, \ldots, r'_{k\alpha_k})$ we have

$$q' \in P_j \quad \Longleftrightarrow \quad p_i^{\gamma_i} \mid q - q' \text{ for each } 1 \leq i \leq k \quad \Longleftrightarrow \quad r_{i,t} = r'_{i,t} \text{ for all } t \leq \gamma_i.$$

Hence $L_j = \{(r'_{11}, \ldots, r'_{k\alpha_k}) \in N : r_{i,t} = r'_{i,t} \text{ for all } t \leq \gamma_i\}$ which means that L_j is a subgrid containing $r(q)$. Moreover, in $L_{j(i)}$, all the coordinates corresponding to p_i are fixed, so it is orthogonal to all of their axes, as desired.

Comment 1. The estimate in the problem is sharp for every n. One of the possible examples is the following one. For each $1 \le i \le k$, $0 \le j \le \alpha_i - 1$, $1 \le k \le p-1$, let

$$P_{i,j,k} = kp_i^j + p_i^{j+1}\mathbb{Z},$$

and add the progression $P_0 = n\mathbb{Z}$. One can easily check that this set satisfies all the problem conditions. There also exist other examples.

On the other hand, the estimate can be adjusted in the following sense. For every $1 \le i \le k$, let $0 = \alpha_{i0}, \alpha_{i1}, \dots, \alpha_{ih_i}$ be all the numbers of the form $\operatorname{ord}_{p_i}(d_j)$ in an increasing order (we delete the repeating occurences of a number, and add a number $0 = \alpha_{i0}$ if it does not occur). Then, repeating the arguments from the solution one can obtain that

$$s \ge 1 + \sum_{i=1}^{k}\sum_{j=1}^{h_i}(p^{\alpha_j - \alpha_{j-1}} - 1).$$

Note that $p^\alpha - 1 \ge \alpha(p-1)$, and the equality is achieved only for $\alpha = 1$. Hence, for reaching the minimal number of the progressions, one should have $\alpha_{i,j} = j$ for all i, j. In other words, for each $1 \le j \le \alpha_i$, there should be an index t such that $\operatorname{ord}_{p_i}(d_t) = j$.

Solution 2. We start with introducing some notation. For positive integer r, we denote $[r] = \{1, 2, \dots, r\}$. Next, we say that a set of progressions $\mathcal{P} = \{P_1, \dots, P_s\}$ *cover* $\mathbb{Z}$ if each integer belongs to some of them; we say that this covering is *minimal* if no proper subset of $\mathcal{P}$ covers $\mathbb{Z}$. Obviously, each covering contains a minimal subcovering.

Next, for a minimal covering $\{P_1, \dots, P_s\}$ and for every $1 \le i \le s$, let d_i be the step of progression P_i, and h_i be some number which is contained in P_i but in none of the other progressions. We assume that $n > 1$, otherwise the problem is trivial. This implies $d_i > 1$, otherwise the progression P_i covers all the numbers, and $n = 1$.

We will prove a more general statement, namely the following

Claim. Assume that the progressions $P_1, \dots, P_s$ and number $n = p_1^{\alpha_1} \dots p_k^{\alpha_k} > 1$ are chosen as in the problem statement. Moreover, choose some nonempty set of indices $I = \{i_1, \dots, i_t\} \subseteq [k]$ and some positive integer $\beta_i \le \alpha_i$ for every $i \in I$. Consider the set of indices

$$T = \left\{ j : 1 \le j \le s, \text{and } p_i^{\alpha_i - \beta_i + 1} \mid d_j \text{ for some } i \in I \right\}.$$

Then

$$|T| \ge 1 + \sum_{i \in I}\beta_i(p_i - 1). \tag{2}$$

Observe that the Claim for $I = [k]$ and $\beta_i = \alpha_i$ implies the problem statement, since the left-hand side in (2) is not greater than s. Hence, it suffices to prove the Claim.

1. First, we prove the Claim assuming that all d_j's are prime numbers. If for some $1 \le i \le k$ we have at least p_i progressions with the step p_i, then they do not intersect and hence cover all the integers; it means that there are no other progressions, and $n = p_i$; the Claim is trivial in this case.

Now assume that for every $1 \le i \le k$, there are not more than $p_i - 1$ progressions with step p_i; each such progression covers the numbers with a fixed residue modulo p_i, therefore there exists a residue $q_i \mod p_i$ which is not touched by these progressions. By the Chinese Remainder Theorem, there exists a number q such that $q \equiv q_i \pmod{p_i}$ for all $1 \le i \le k$; this number cannot be covered by any progression with step p_i, hence it is not covered at all. A contradiction.

2. Now, we assume that the general Claim is not valid, and hence we consider a counterexample $\{P_1, \dots, P_s\}$ for the Claim; we can choose it to be minimal in the following sense:

• the number n is minimal possible among all the counterexamples;

• the sum $\sum_i d_i$ is minimal possible among all the counterexamples having the chosen value of n.

As was mentioned above, not all numbers d_i are primes; hence we can assume that d_1 is composite, say $p_1 \mid d_1$ and $d_1' = \frac{d_1}{p_1} > 1$. Consider a progression P_1' having the step d_1', and containing P_1. We will focus on two coverings constructed as follows.

(i) Surely, the progressions $P_1', P_2, \dots, P_s$ cover $\mathbb{Z}$, though this covering in not necessarily minimal. So, choose some minimal subcovering $\mathcal{P}'$ in it; surely $P_1' \in \mathcal{P}'$ since h_1 is not covered by $P_2, \dots, P_s$, so we may assume that $\mathcal{P}' = \{P_1', P_2, \dots, P_{s'}\}$ for some $s' \le s$. Furthermore, the period of the covering $\mathcal{P}'$ can appear to be less than n; so we denote this period by

$$n' = p_1^{\alpha_1-\sigma_1} \dots p_k^{\alpha_k-\sigma_k} = \text{l.c.m.}\big(d_1', d_2, \dots, d_{s'}\big).$$

Observe that for each $P_j \notin \mathcal{P}'$, we have $h_j \in P_1'$, otherwise h_j would not be covered by $\mathcal{P}$.

(ii) On the other hand, each nonempty set of the form $R_i = P_i \cap P_1'$ $(1 \le i \le s)$ is also a progression with a step $r_i = \text{l.c.m.}(d_i, d_1')$, and such sets cover P_1'. Scaling these progressions with the ratio $1/d_1'$, we obtain the progressions Q_i with steps $q_i = r_i/d_1'$ which cover $\mathbb{Z}$. Now we choose a minimal subcovering $\mathcal{Q}$ of this covering; again we should have $Q_1 \in \mathcal{Q}$ by the reasons of h_1. Now, denote the period of $\mathcal{Q}$ by

$$n'' = \text{l.c.m.}\{q_i : Q_i \in \mathcal{Q}\} = \frac{\text{l.c.m.}\{r_i : Q_i \in \mathcal{Q}\}}{d_1'} = \frac{p_1^{\gamma_1} \dots p_k^{\gamma_k}}{d_1'}.$$

Note that if $h_j \in P_1'$, then the image of h_j under the scaling can be covered by Q_j only; so, in this case we have $Q_j \in \mathcal{Q}$.

Our aim is to find the desired number of progressions in coverings $\mathcal{P}$ and $\mathcal{Q}$. First, we have $n \ge n'$, and the sum of the steps in $\mathcal{P}'$ is less than that in $\mathcal{P}$; hence the Claim is valid for $\mathcal{P}'$. We apply it to the set of indices $I' = \{i \in I : \beta_i > \sigma_i\}$ and the exponents $\beta_i' = \beta_i - \sigma_i$; hence the set under consideration is

$$T' = \Big\{ j : 1 \le j \le s', \text{and } p_i^{(\alpha_i-\sigma_i)-\beta_i'+1} = p_i^{\alpha_i-\beta_i+1} \mid d_j \text{ for some } i \in I' \Big\} \subseteq T \cap [s'],$$

and we obtain that

$$|T \cap [s']| \ge |T'| \ge 1 + \sum_{i \in I'} (\beta_i - \sigma_i)(p_i - 1) = 1 + \sum_{i \in I} (\beta_i - \sigma_i)_+ (p_i - 1),$$

where $(x)_+ = \max\{x, 0\}$; the latter equality holds as for $i \notin I'$ we have $\beta_i \le \sigma_i$.

Observe that $x = (x-y)_+ + \min\{x, y\}$ for all x, y. So, if we find at least

$$G = \sum_{i \in I} \min\{\beta_i, \sigma_i\}(p_i - 1)$$

indices in $T \cap \{s'+1, \dots, s\}$, then we would have

$$|T| = |T \cap [s']| + |T \cap \{s'+1, \dots, s\}| \ge 1 + \sum_{i \in I} \big((\beta_i - \sigma_i)_+ + \min\{\beta_i, \sigma_i\}\big)(p_i - 1) = 1 + \sum_{i \in I} \beta_i(p_i - 1),$$

thus leading to a contradiction with the choice of $\mathcal{P}$. We will find those indices among the indices of progressions in $\mathcal{Q}$.

3. Now denote $I'' = \{i \in I : \sigma_i > 0\}$ and consider some $i \in I''$; then $p_i^{\alpha_i} \nmid n'$. On the other hand, there exists an index $j(i)$ such that $p_i^{\alpha_i} \mid d_{j(i)}$; this means that $d_{j(i)} \nmid n'$ and hence $P_{j(i)}$ cannot appear in $\mathcal{P}'$, so $j(i) > s'$. Moreover, we have observed before that in this case $h_{j(i)} \in P'_1$, hence $Q_{j(i)} \in \mathcal{Q}$. This means that $q_{j(i)} \mid n''$, therefore $\gamma_i = \alpha_i$ for each $i \in I''$ (recall here that $q_i = r_i/d'_1$ and hence $d_{j(i)} \mid r_{j(i)} \mid d'_1 n''$).

Let $d'_1 = p_1^{\tau_1} \dots p_k^{\tau_k}$. Then $n'' = p_1^{\gamma_1-\tau_1} \dots p_k^{\gamma_i-\tau_i}$. Now, if $i \in I''$, then for every β the condition $p_i^{(\gamma_i-\tau_i)-\beta+1} \mid q_j$ is equivalent to $p_i^{\alpha_i-\beta+1} \mid r_j$.

Note that $n'' \leq n/d'_1 < n$, hence we can apply the Claim to the covering $\mathcal{Q}$. We perform this with the set of indices I'' and the exponents $\beta''_i = \min\{\beta_i, \sigma_i\} > 0$. So, the set under consideration is

$$T'' = \left\{j : Q_j \in \mathcal{Q}, \text{and } p_i^{(\gamma_i-\tau_i)-\min\{\beta_i,\sigma_i\}+1} \mid q_j \text{ for some } i \in I''\right\}$$
$$= \left\{j : Q_j \in \mathcal{Q}, \text{and } p_i^{\alpha_i-\min\{\beta_i,\sigma_i\}+1} \mid r_j \text{ for some } i \in I''\right\},$$

and we obtain $|T''| \geq 1 + G$. Finally, we claim that $T'' \subseteq T \cap \big(\{1\} \cup \{s'+1, \dots, s\}\big)$; then we will obtain $|T \cap \{s'+1, \dots, s\}| \geq G$, which is exactly what we need.

To prove this, consider any $j \in T''$. Observe first that $\alpha_i - \min\{\beta_i, \sigma_i\} + 1 > \alpha_i - \sigma_i \geq \tau_i$, hence from $p_i^{\alpha_i-\min\{\beta_i,\sigma_i\}+1} \mid r_j = \text{l.c.m.}(d'_1, d_j)$ we have $p_i^{\alpha_i-\min\{\beta_i,\sigma_i\}+1} \mid d_j$, which means that $j \in T$. Next, the exponent of p_i in d_j is greater than that in n', which means that $P_j \notin \mathcal{P}'$. This may appear only if $j = 1$ or $j > s'$, as desired. This completes the proof.

Comment 2. A grid analogue of the Claim is also valid. It reads as following.

Claim. Assume that the grid N is covered by subgrids $L_1, L_2, \dots, L_s$ so that

(ii′) each subgrid contains a point which is not covered by other subgrids;

(iii) for each coordinate axis, there exists a subgrid L_i orthogonal to this axis.

Choose some set of indices $I = \{i_1, \dots, i_t\} \subset [k]$, and consider the set of indices

$$T = \{j : 1 \leq j \leq s, \text{and } L_j \text{ is orthogonal to the } i\text{th axis for some } i \in I\}.$$

Then
$$|T| \geq 1 + \sum_{i \in I}(n_i - 1).$$

This Claim may be proved almost in the same way as in Solution 1.

Geometry

G1. Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

(United Kingdom)

Solution 1. The line EF intersects the circumcircle at two points. Depending on the choice of P, there are two different cases to consider.

Case 1: The point P lies on the ray EF (see Fig. 1).

Let $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. The quadrilaterals $BCEF$ and $CAFD$ are cyclic due to the right angles at D, E and F. So,

$$\angle BDF = 180^\circ - \angle FDC = \angle CAF = \alpha,$$
$$\angle AFE = 180^\circ - \angle EFB = \angle BCE = \gamma,$$
$$\angle DFB = 180^\circ - \angle AFD = \angle DCA = \gamma.$$

Since P lies on the arc AB of the circumcircle, $\angle PBA < \angle BCA = \gamma$. Hence, we have

$$\angle PBD + \angle BDF = \angle PBA + \angle ABD + \angle BDF < \gamma + \beta + \alpha = 180^\circ,$$

and the point Q must lie on the extensions of BP and DF beyond the points P and F, respectively.

From the cyclic quadrilateral $APBC$ we get

$$\angle QPA = 180^\circ - \angle APB = \angle BCA = \gamma = \angle DFB = \angle QFA.$$

Hence, the quadrilateral $AQPF$ is cyclic. Then $\angle AQP = 180^\circ - \angle PFA = \angle AFE = \gamma$.

We obtained that $\angle AQP = \angle QPA = \gamma$, so the triangle AQP is isosceles, $AP = AQ$.

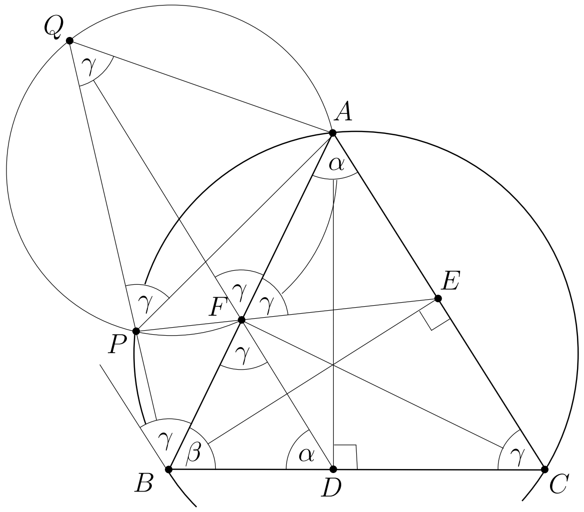


Fig. 1

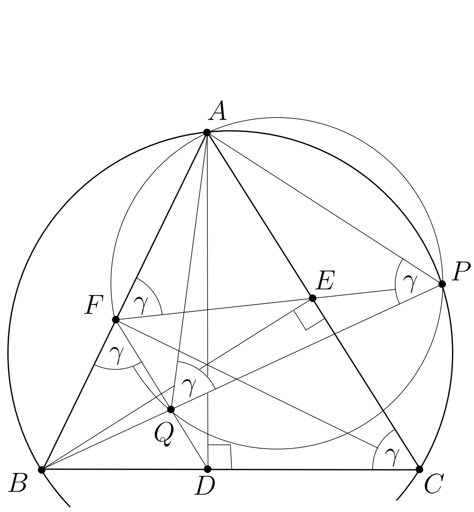


Fig. 2

Case 2: The point P lies on the ray FE (see Fig. 2). In this case the point Q lies inside the segment FD.

Similarly to the first case, we have

$$\angle QPA = \angle BCA = \gamma = \angle DFB = 180^\circ - \angle AFQ.$$

Hence, the quadrilateral $AFQP$ is cyclic.

Then $\angle AQP = \angle AFP = \angle AFE = \gamma = \angle QPA$. The triangle AQP is isosceles again, $\angle AQP = \angle QPA$ and thus $AP = AQ$.

Comment. Using signed angles, the two possible configurations can be handled simultaneously, without investigating the possible locations of P and Q.

Solution 2. For arbitrary points X, Y on the circumcircle, denote by $\overset{\frown}{XY}$ the central angle of the arc XY.

Let P and P' be the two points where the line EF meets the circumcircle; let P lie on the arc AB and let P' lie on the arc CA. Let BP and BP' meet the line DF and Q and Q', respectively (see Fig. 3). We will prove that $AP = AP' = AQ = AQ'$.

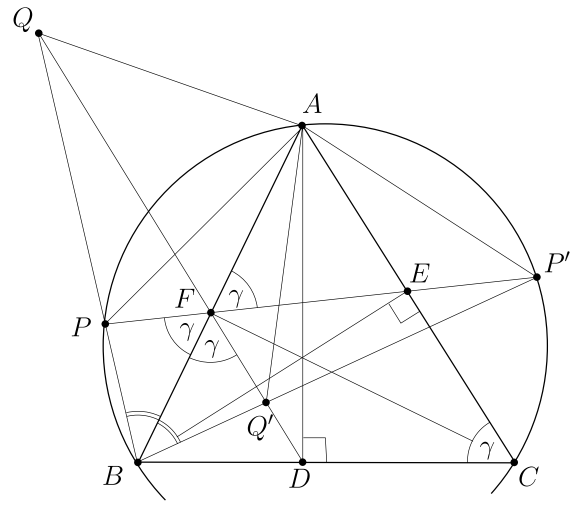


Fig. 3

Like in the first solution, we have $\angle AFE = \angle BFP = \angle DFB = \angle BCA = \gamma$ from the cyclic quadrilaterals $BCEF$ and $CAFD$.

By $\overset{\frown}{PB} + \overset{\frown}{P'A} = 2\angle AFP' = 2\gamma = 2\angle BCA = \overset{\frown}{AP} + \overset{\frown}{PB}$, we have

$$\overset{\frown}{AP} = \overset{\frown}{P'A}, \quad \angle PBA = \angle ABP' \qquad \text{and} \quad AP = AP'. \tag{1}$$

Due to $\overset{\frown}{AP} = \overset{\frown}{P'A}$, the lines BP and BQ' are symmetrical about line AB.

Similarly, by $\angle BFP = \angle Q'FB$, the lines FP and FQ' are symmetrical about AB. It follows that also the points P and P' are symmetrical to Q' and Q, respectively. Therefore,

$$AP = AQ' \quad \text{and} \quad AP' = AQ. \tag{2}$$

The relations (1) and (2) together prove $AP = AP' = AQ = AQ'$.

G2. Point P lies inside triangle ABC. Lines AP, BP, CP meet the circumcircle of ABC again at points K, L, M, respectively. The tangent to the circumcircle at C meets line AB at S. Prove that $SC = SP$ if and only if $MK = ML$.

(Poland)

Solution 1. We assume that $CA > CB$, so point S lies on the ray AB.

From the similar triangles $\triangle PKM \sim \triangle PCA$ and $\triangle PLM \sim \triangle PCB$ we get $\dfrac{PM}{KM} = \dfrac{PA}{CA}$ and $\dfrac{LM}{PM} = \dfrac{CB}{PB}$. Multiplying these two equalities, we get

$$\frac{LM}{KM} = \frac{CB}{CA} \cdot \frac{PA}{PB}.$$

Hence, the relation $MK = ML$ is equivalent to $\dfrac{CB}{CA} = \dfrac{PB}{PA}$.

Denote by E the foot of the bisector of angle B in triangle ABC. Recall that the locus of points X for which $\dfrac{XA}{XB} = \dfrac{CA}{CB}$ is the Apollonius circle Ω with the center Q on the line AB, and this circle passes through C and E. Hence, we have $MK = ML$ if and only if P lies on Ω, that is $QP = QC$.

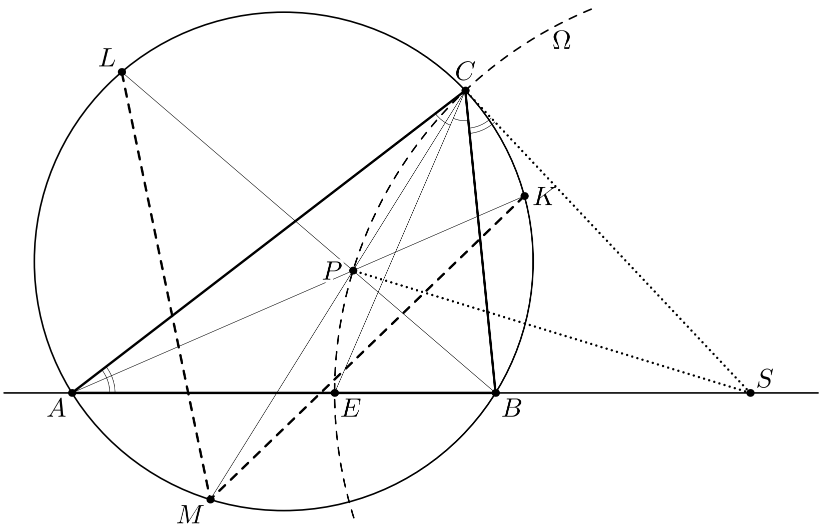


Fig. 1

Now we prove that $S = Q$, thus establishing the problem statement. We have $\angle CES = \angle CAE + \angle ACE = \angle BCS + \angle ECB = \angle ECS$, so $SC = SE$. Hence, the point S lies on AB as well as on the perpendicular bisector of CE and therefore coincides with Q.

Solution 2. As in the previous solution, we assume that S lies on the ray AB.

1. Let P be an arbitrary point inside both the circumcircle ω of the triangle ABC and the angle ASC, the points K, L, M defined as in the problem. We claim that $SP = SC$ implies $MK = ML$.

Let E and F be the points of intersection of the line SP with ω, point E lying on the segment SP (see Fig. 2).

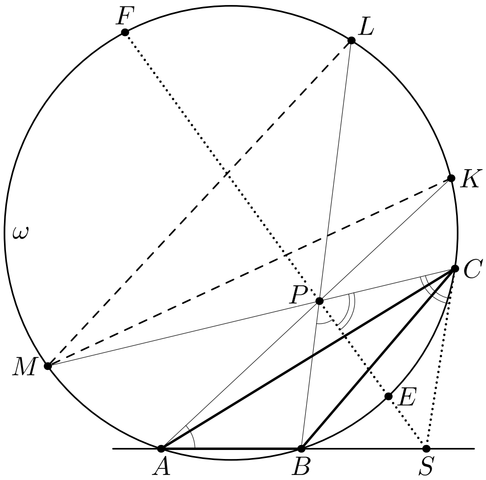


Fig. 2

We have $SP^2 = SC^2 = SA \cdot SB$, so $\dfrac{SP}{SB} = \dfrac{SA}{SP}$, and hence $\triangle PSA \sim \triangle BSP$. Then $\angle BPS = \angle SAP$. Since $2\angle BPS = \overset{\frown}{BE} + \overset{\frown}{LF}$ and $2\angle SAP = \overset{\frown}{BE} + \overset{\frown}{EK}$ we have

$$\overset{\frown}{LF} = \overset{\frown}{EK}. \tag{1}$$

On the other hand, from $\angle SPC = \angle SCP$ we have $\overset{\frown}{EC} + \overset{\frown}{MF} = \overset{\frown}{EC} + \overset{\frown}{EM}$, or

$$\overset{\frown}{MF} = \overset{\frown}{EM}. \tag{2}$$

From (1) and (2) we get $\overset{\frown}{MFL} = \overset{\frown}{MF} + \overset{\frown}{FL} = \overset{\frown}{ME} + \overset{\frown}{EK} = \overset{\frown}{MEK}$ and hence $MK = ML$. The claim is proved.

2. We are left to prove the converse. So, assume that $MK = ML$, and introduce the points E and F as above. We have $SC^2 = SE \cdot SF$; hence, there exists a point P' lying on the segment EF such that $SP' = SC$ (see Fig. 3).

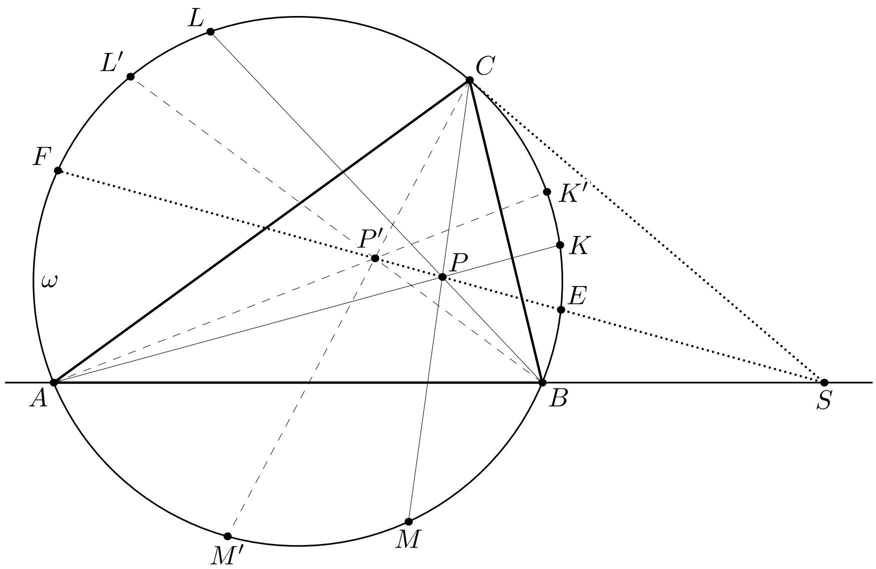


Fig. 3

Assume that $P \neq P'$. Let the lines AP', BP', CP' meet ω again at points K', L', M' respectively. Now, if P' lies on the segment PF then by the first part of the solution we have $\widehat{M'FL'} = \widehat{M'EK'}$. On the other hand, we have $\widehat{MFL} > \widehat{M'FL'} = \widehat{M'EK'} > \widehat{MEK}$, therefore $\widehat{MFL} > \widehat{MEK}$ which contradicts $MK = ML$.

Similarly, if point P' lies on the segment EP then we get $\widehat{MFL} < \widehat{MEK}$ which is impossible. Therefore, the points P and P' coincide and hence $SP = SP' = SC$.

Solution 3. We present a different proof of the converse direction, that is, $MK = ML \Rightarrow SP = SC$. As in the previous solutions we assume that $CA > CB$, and the line SP meets ω at E and F.

From $ML = MK$ we get $\widehat{MEK} = \widehat{MFL}$. Now we claim that $\widehat{ME} = \widehat{MF}$ and $\widehat{EK} = \widehat{FL}$.

To the contrary, suppose first that $\widehat{ME} > \widehat{MF}$; then $\widehat{EK} = \widehat{MEK} - \widehat{ME} < \widehat{MFL} - \widehat{MF} = \widehat{FL}$. Now, the inequality $\widehat{ME} > \widehat{MF}$ implies $2\angle SCM = \widehat{EC} + \widehat{ME} > \widehat{EC} + \widehat{MF} = 2\angle SPC$ and hence $SP > SC$. On the other hand, the inequality $\widehat{EK} < \widehat{FL}$ implies $2\angle SPK = \widehat{EK} + \widehat{AF} < \widehat{FL} + \widehat{AF} = 2\angle ABL$, hence

$$\angle SPA = 180^\circ - \angle SPK > 180^\circ - \angle ABL = \angle SBP.$$

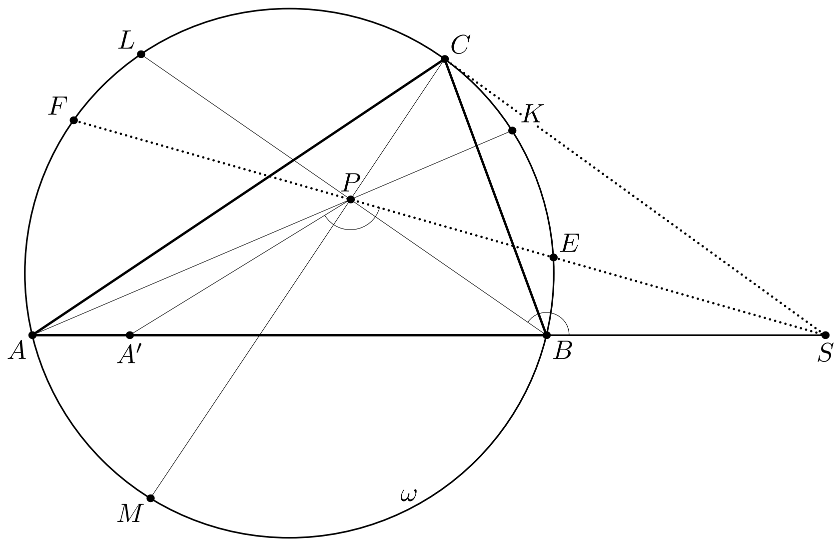


Fig. 4

Consider the point A' on the ray SA for which $\angle SPA' = \angle SBP$; in our case, this point lies on the segment SA (see Fig. 4). Then $\triangle SBP \sim \triangle SPA'$ and $SP^2 = SB \cdot SA' < SB \cdot SA = SC^2$. Therefore, $SP < SC$ which contradicts $SP > SC$.

Similarly, one can prove that the inequality $\widehat{ME} < \widehat{MF}$ is also impossible. So, we get $\widehat{ME} = \widehat{MF}$ and therefore $2\angle SCM = \widehat{EC} + \widehat{ME} = \widehat{EC} + \widehat{MF} = 2\angle SPC$, which implies $SC = SP$.

G3. Let $A_1A_2\dots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections $P_1, \dots, P_n$ onto lines $A_1A_2, \dots, A_nA_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1, \dots, X_n$ on sides $A_1A_2, \dots, A_nA_1$ respectively,

$$\max\left\{\frac{X_1X_2}{P_1P_2}, \dots, \frac{X_nX_1}{P_nP_1}\right\} \geq 1.$$

(Armenia)

Solution 1. Denote $P_{n+1} = P_1$, $X_{n+1} = X_1$, $A_{n+1} = A_1$.

Lemma. Let point Q lies inside $A_1A_2\dots A_n$. Then it is contained in at least one of the circumcircles of triangles $X_1A_2X_2, \dots, X_nA_1X_1$.

Proof. If Q lies in one of the triangles $X_1A_2X_2, \dots, X_nA_1X_1$, the claim is obvious. Otherwise Q lies inside the polygon $X_1X_2\dots X_n$ (see Fig. 1). Then we have

$$\begin{aligned}(\angle X_1A_2X_2 + \angle X_1QX_2) + \dots + (\angle X_nA_1X_1 + \angle X_nQX_1)\\ = (\angle X_1A_1X_2 + \dots + \angle X_nA_1X_1) + (\angle X_1QX_2 + \dots + \angle X_nQX_1) = (n-2)\pi + 2\pi = n\pi,\end{aligned}$$

hence there exists an index i such that $\angle X_iA_{i+1}X_{i+1} + \angle X_iQX_{i+1} \geq \frac{\pi n}{n} = \pi$. Since the quadrilateral $QX_iA_{i+1}X_{i+1}$ is convex, this means exactly that Q is contained the circumcircle of $\triangle X_iA_{i+1}X_{i+1}$, as desired. $\square$

Now we turn to the solution. Applying lemma, we get that P lies inside the circumcircle of triangle $X_iA_{i+1}X_{i+1}$ for some i. Consider the circumcircles ω and Ω of triangles $P_iA_{i+1}P_{i+1}$ and $X_iA_{i+1}X_{i+1}$ respectively (see Fig. 2); let r and R be their radii. Then we get $2r = A_{i+1}P \leq 2R$ (since P lies inside Ω), hence

$$P_iP_{i+1} = 2r\sin\angle P_iA_{i+1}P_{i+1} \leq 2R\sin\angle X_iA_{i+1}X_{i+1} = X_iX_{i+1},$$

QED.

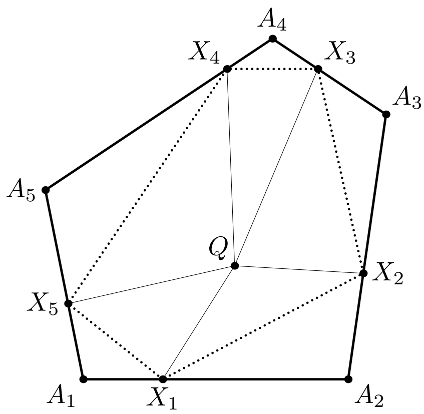


Fig. 1

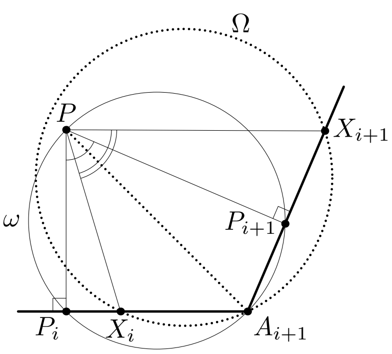


Fig. 2

Solution 2. As in Solution 1, we assume that all indices of points are considered modulo n.

We will prove a bit stronger inequality, namely

$$\max\left\{\frac{X_1X_2}{P_1P_2}\cos\alpha_1,\dots,\frac{X_nX_1}{P_nP_1}\cos\alpha_n\right\}\ge 1,$$

where α_i $(1\le i\le n)$ is the angle between lines X_iX_{i+1} and P_iP_{i+1}. We denote $\beta_i=\angle A_iP_iP_{i-1}$ and $\gamma_i=\angle A_{i+1}P_iP_{i+1}$ for all $1\le i\le n$.

Suppose that for some $1\le i\le n$, point X_i lies on the segment A_iP_i, while point X_{i+1} lies on the segment $P_{i+1}A_{i+2}$. Then the projection of the segment X_iX_{i+1} onto the line P_iP_{i+1} contains segment P_iP_{i+1}, since γ_i and β_{i+1} are acute angles (see Fig. 3). Therefore, $X_iX_{i+1}\cos\alpha_i\ge P_iP_{i+1}$, and in this case the statement is proved.

So, the only case left is when point X_i lies on segment P_iA_{i+1} for all $1\le i\le n$ (the case when each X_i lies on segment A_iP_i is completely analogous).

Now, assume to the contrary that the inequality

$$X_iX_{i+1}\cos\alpha_i<P_iP_{i+1} \tag{1}$$

holds for every $1\le i\le n$. Let Y_i and Y'_{i+1} be the projections of X_i and X_{i+1} onto P_iP_{i+1}. Then inequality (1) means exactly that $Y_iY'_{i+1}<P_iP_{i+1}$, or $P_iY_i>P_{i+1}Y'_{i+1}$ (again since γ_i and β_{i+1} are acute; see Fig. 4). Hence, we have

$$X_iP_i\cos\gamma_i>X_{i+1}P_{i+1}\cos\beta_{i+1},\qquad 1\le i\le n.$$

Multiplying these inequalities, we get

$$\cos\gamma_1\cos\gamma_2\cdots\cos\gamma_n>\cos\beta_1\cos\beta_2\cdots\cos\beta_n. \tag{2}$$

On the other hand, the sines theorem applied to triangle PP_iP_{i+1} provides

$$\frac{PP_i}{PP_{i+1}}=\frac{\sin\left(\frac{\pi}{2}-\beta_{i+1}\right)}{\sin\left(\frac{\pi}{2}-\gamma_i\right)}=\frac{\cos\beta_{i+1}}{\cos\gamma_i}.$$

Multiplying these equalities we get

$$1=\frac{\cos\beta_2}{\cos\gamma_1}\cdot\frac{\cos\beta_3}{\cos\gamma_2}\cdots\frac{\cos\beta_1}{\cos\gamma_n}$$

which contradicts (2).

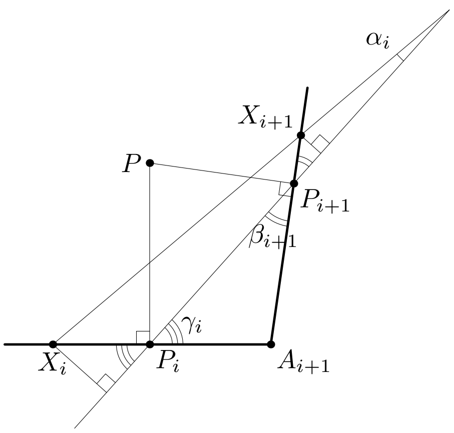


Fig. 3

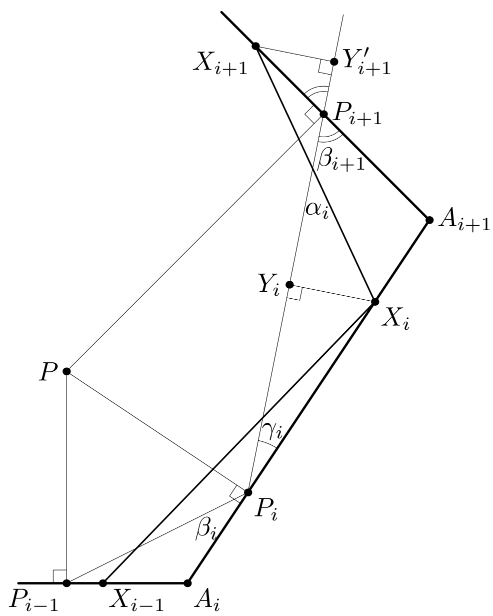


Fig. 4

G4. Let I be the incenter of a triangle ABC and Γ be its circumcircle. Let the line AI intersect Γ at a point $D \neq A$. Let F and E be points on side BC and arc BDC respectively such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. Finally, let G be the midpoint of the segment IF. Prove that the lines DG and EI intersect on Γ.

(Hong Kong)

Solution 1. Let X be the second point of intersection of line EI with Γ, and L be the foot of the bisector of angle BAC. Let G' and T be the points of intersection of segment DX with lines IF and AF, respectively. We are to prove that $G = G'$, or $IG' = G'F$. By the Menelaus theorem applied to triangle AIF and line DX, it means that we need the relation

$$1 = \frac{G'F}{IG'} = \frac{TF}{AT} \cdot \frac{AD}{ID}, \qquad \text{or} \quad \frac{TF}{AT} = \frac{ID}{AD}.$$

Let the line AF intersect Γ at point $K \neq A$ (see Fig. 1); since $\angle BAK = \angle CAE$ we have $\widehat{BK} = \widehat{CE}$, hence $KE \parallel BC$. Notice that $\angle IAT = \angle DAK = \angle EAD = \angle EXD = \angle IXT$, so the points I, A, X, T are concyclic. Hence we have $\angle ITA = \angle IXA = \angle EXA = \angle EKA$, so $IT \parallel KE \parallel BC$. Therefore we obtain $\dfrac{TF}{AT} = \dfrac{IL}{AI}$.

Since CI is the bisector of $\angle ACL$, we get $\dfrac{IL}{AI} = \dfrac{CL}{AC}$. Furthermore, $\angle DCL = \angle DCB = \angle DAB = \angle CAD = \frac{1}{2}\angle BAC$, hence the triangles DCL and DAC are similar; therefore we get $\dfrac{CL}{AC} = \dfrac{DC}{AD}$. Finally, it is known that the midpoint D of arc BC is equidistant from points I, B, C, hence $\dfrac{DC}{AD} = \dfrac{ID}{AD}$.

Summarizing all these equalities, we get

$$\frac{TF}{AT} = \frac{IL}{AI} = \frac{CL}{AC} = \frac{DC}{AD} = \frac{ID}{AD},$$

as desired.

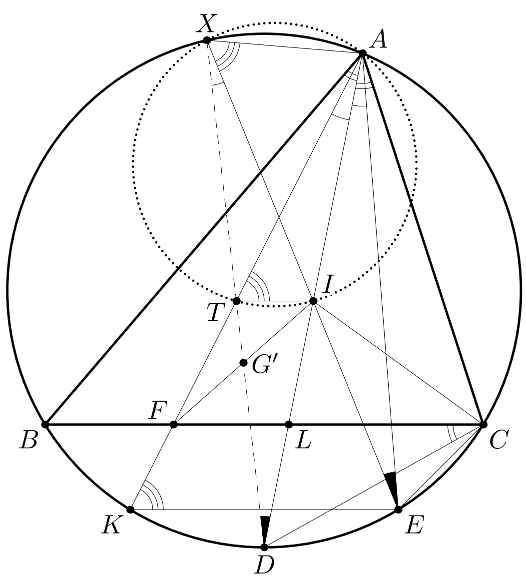


Fig. 1

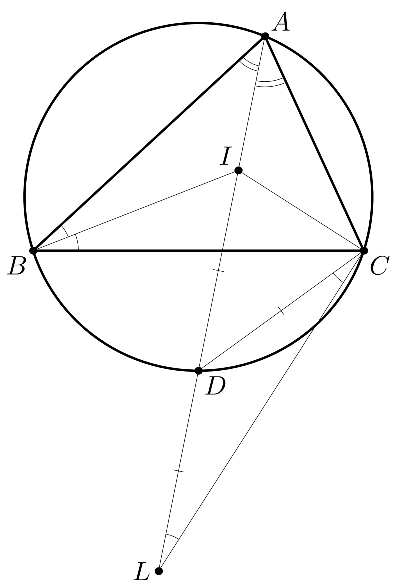


Fig. 2

Comment. The equality $\dfrac{AI}{IL} = \dfrac{AD}{DI}$ is known and can be obtained in many different ways. For instance, one can consider the inversion with center D and radius $DC = DI$. This inversion takes $\widehat{BAC}$ to the segment BC, so point A goes to L. Hence $\dfrac{IL}{DI} = \dfrac{AI}{AD}$, which is the desired equality.

Solution 2. As in the previous solution, we introduce the points X, T and K and note that it suffice to prove the equality

$$\frac{TF}{AT} = \frac{DI}{AD} \quad \Longleftrightarrow \quad \frac{TF + AT}{AT} = \frac{DI + AD}{AD} \quad \Longleftrightarrow \quad \frac{AT}{AD} = \frac{AF}{DI + AD}.$$

Since $\angle FAD = \angle EAI$ and $\angle TDA = \angle XDA = \angle XEA = \angle IEA$, we get that the triangles ATD and AIE are similar, therefore $\dfrac{AT}{AD} = \dfrac{AI}{AE}$.

Next, we also use the relation $DB = DC = DI$. Let J be the point on the extension of segment AD over point D such that $DJ = DI = DC$ (see Fig. 2). Then $\angle DJC = \angle JCD = \frac{1}{2}(\pi - \angle JDC) = \frac{1}{2}\angle ADC = \frac{1}{2}\angle ABC = \angle ABI$. Moreover, $\angle BAI = \angle JAC$, hence triangles ABI and AJC are similar, so $\dfrac{AB}{AJ} = \dfrac{AI}{AC}$, or $AB \cdot AC = AJ \cdot AI = (DI + AD) \cdot AI$.

On the other hand, we get $\angle ABF = \angle ABC = \angle AEC$ and $\angle BAF = \angle CAE$, so triangles ABF and AEC are also similar, which implies $\dfrac{AF}{AC} = \dfrac{AB}{AE}$, or $AB \cdot AC = AF \cdot AE$.

Summarizing we get

$$(DI + AD) \cdot AI = AB \cdot AC = AF \cdot AE \quad \Rightarrow \quad \frac{AI}{AE} = \frac{AF}{AD + DI} \quad \Rightarrow \quad \frac{AT}{AD} = \frac{AF}{AD + DI},$$

as desired.

Comment. In fact, point J is an excenter of triangle ABC.

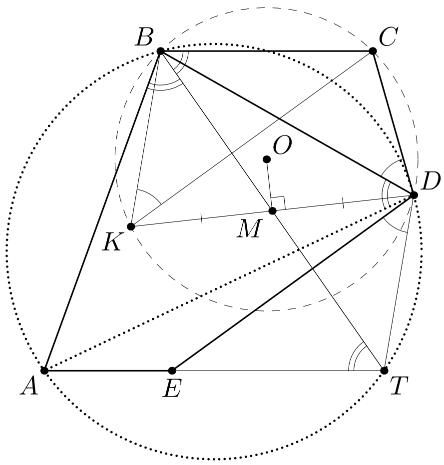
G5. Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90°$, prove that $2\angle BDA = \angle CDE$.

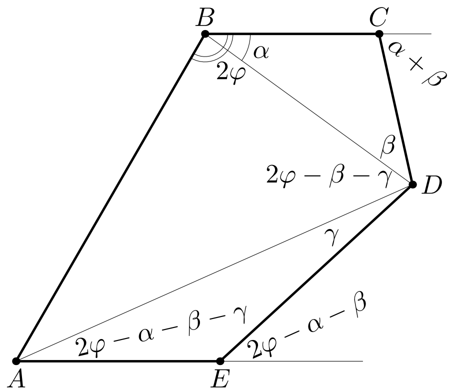
(Ukraine)

Solution 1. Choose point T on ray AE such that $AT = AB$; then from $AE \parallel BC$ we have $\angle CBT = \angle ATB = \angle ABT$, so BT is the bisector of $\angle ABC$. On the other hand, we have $ET = AT - AE = AB - AE = BC$, hence quadrilateral $BCTE$ is a parallelogram, and the midpoint M of its diagonal CE is also the midpoint of the other diagonal BT.

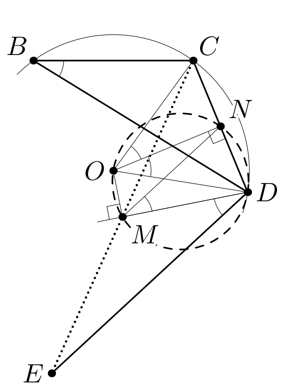
Next, let point K be symmetrical to D with respect to M. Then OM is the perpendicular bisector of segment DK, and hence $OD = OK$, which means that point K lies on the circumcircle of triangle BCD. Hence we have $\angle BDC = \angle BKC$. On the other hand, the angles BKC and TDE are symmetrical with respect to M, so $\angle TDE = \angle BKC = \angle BDC$.

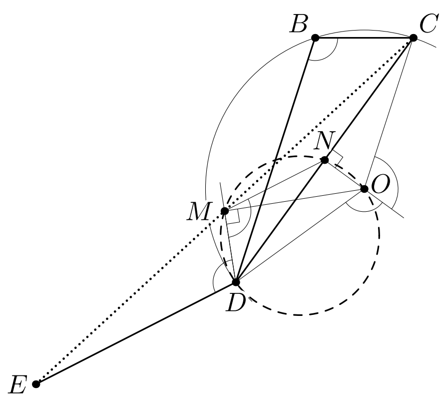
Therefore, $\angle BDT = \angle BDE + \angle EDT = \angle BDE + \angle BDC = \angle CDE = \angle ABC = 180° - \angle BAT$. This means that the points A, B, D, T are concyclic, and hence $\angle ADB = \angle ATB = \frac{1}{2}\angle ABC = \frac{1}{2}\angle CDE$, as desired.





Solution 2. Let$\angle CBD = \alpha$, $\angle BDC = \beta$, $\angle ADE = \gamma$, and $\angle ABC = \angle CDE = 2\varphi$. Then we have $\angle ADB = 2\varphi - \beta - \gamma$, $\angle BCD = 180° - \alpha - \beta$, $\angle AED = 360° - \angle BCD - \angle CDE = 180° - 2\varphi + \alpha + \beta$, and finally $\angle DAE = 180° - \angle ADE - \angle AED = 2\varphi - \alpha - \beta - \gamma$.





Let N be the midpoint of CD; then $\angle DNO = 90° = \angle DMO$, hence points M, N lie on the circle with diameter OD. Now, if points O and M lie on the same side of CD, we have $\angle DMN = \angle DON = \frac{1}{2}\angle DOC = \alpha$; in the other case, we have $\angle DMN = 180° - \angle DON = \alpha$;

so, in both cases $\angle DMN = \alpha$ (see Figures). Next, since MN is a midline in triangle CDE, we have $\angle MDE = \angle DMN = \alpha$ and $\angle NDM = 2\varphi - \alpha$.

Now we apply the sine rule to the triangles ABD, ADE (twice), BCD and MND obtaining

$$\frac{AB}{AD} = \frac{\sin(2\varphi - \beta - \gamma)}{\sin(2\varphi - \alpha)}, \quad \frac{AE}{AD} = \frac{\sin\gamma}{\sin(2\varphi - \alpha - \beta)}, \quad \frac{DE}{AD} = \frac{\sin(2\varphi - \alpha - \beta - \gamma)}{\sin(2\varphi - \alpha - \beta)},$$
$$\frac{BC}{CD} = \frac{\sin\beta}{\sin\alpha}, \quad \frac{CD}{DE} = \frac{CD/2}{DE/2} = \frac{ND}{NM} = \frac{\sin\alpha}{\sin(2\varphi - \alpha)},$$

which implies

$$\frac{BC}{AD} = \frac{BC}{CD} \cdot \frac{CD}{DE} \cdot \frac{DE}{AD} = \frac{\sin\beta \cdot \sin(2\varphi - \alpha - \beta - \gamma)}{\sin(2\varphi - \alpha) \cdot \sin(2\varphi - \alpha - \beta)}.$$

Hence, the condition $AB = AE + BC$, or equivalently $\dfrac{AB}{AD} = \dfrac{AE + BC}{AD}$, after multiplying by the common denominator rewrites as

$$\begin{aligned}
&\sin(2\varphi - \alpha - \beta) \cdot \sin(2\varphi - \beta - \gamma) = \sin\gamma \cdot \sin(2\varphi - \alpha) + \sin\beta \cdot \sin(2\varphi - \alpha - \beta - \gamma)\\
\iff\ &\cos(\gamma - \alpha) - \cos(4\varphi - 2\beta - \alpha - \gamma) = \cos(2\varphi - \alpha - 2\beta - \gamma) - \cos(2\varphi + \gamma - \alpha)\\
\iff\ &\cos(\gamma - \alpha) + \cos(2\varphi + \gamma - \alpha) = \cos(2\varphi - \alpha - 2\beta - \gamma) + \cos(4\varphi - 2\beta - \alpha - \gamma)\\
\iff\ &\cos\varphi \cdot \cos(\varphi + \gamma - \alpha) = \cos\varphi \cdot \cos(3\varphi - 2\beta - \alpha - \gamma)\\
\iff\ &\cos\varphi \cdot \big(\cos(\varphi + \gamma - \alpha) - \cos(3\varphi - 2\beta - \alpha - \gamma)\big) = 0\\
\iff\ &\cos\varphi \cdot \sin(2\varphi - \beta - \alpha) \cdot \sin(\varphi - \beta - \gamma) = 0.
\end{aligned}$$

Since $2\varphi - \beta - \alpha = 180^\circ - \angle AED < 180^\circ$ and $\varphi = \frac{1}{2}\angle ABC < 90^\circ$, it follows that $\varphi = \beta + \gamma$, hence $\angle BDA = 2\varphi - \beta - \gamma = \varphi = \frac{1}{2}\angle CDE$, as desired.

G6. The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of an acute-angled triangle ABC. Prove that the incenter of triangle ABC lies inside triangle XYZ.

G6′. The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of a triangle ABC. Prove that if the incenter of triangle ABC lies outside triangle XYZ, then one of the angles of triangle ABC is greater than $120°$.

(Bulgaria)

Solution 1 for G6. We will prove a stronger fact; namely, we will show that the incenter I of triangle ABC lies inside the incircle of triangle XYZ (and hence surely inside triangle XYZ itself). We denote by $d(U, VW)$ the distance between point U and line VW.

Denote by O the incenter of $\triangle XYZ$ and by r, r' and R' the inradii of triangles ABC, XYZ and the circumradius of XYZ, respectively. Then we have $R' = 2r'$, and the desired inequality is $OI \le r'$. We assume that $O \ne I$; otherwise the claim is trivial.

Let the incircle of $\triangle ABC$ touch its sides BC, AC, AB at points A_1, B_1, C_1 respectively. The lines IA_1, IB_1, IC_1 cut the plane into 6 acute angles, each one containing one of the points A_1, B_1, C_1 on its border. We may assume that O lies in an angle defined by lines IA_1, IC_1 and containing point C_1 (see Fig. 1). Let A' and C' be the projections of O onto lines IA_1 and IC_1, respectively.

Since $OX = R'$, we have $d(O, BC) \le R'$. Since $OA' \parallel BC$, it follows that $d(A', BC) = A'I + r \le R'$, or $A'I \le R' - r$. On the other hand, the incircle of $\triangle XYZ$ lies inside $\triangle ABC$, hence $d(O, AB) \ge r'$, and analogously we get $d(O, AB) = C'C_1 = r - IC' \ge r'$, or $IC' \le r - r'$.

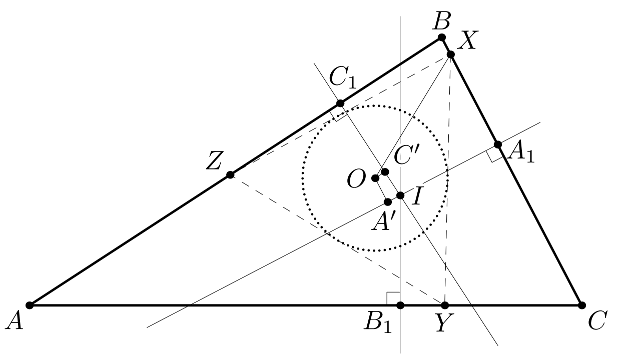


Fig. 1

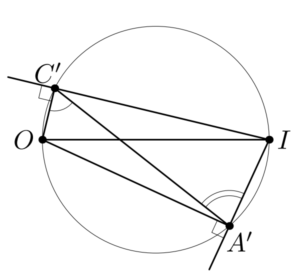


Fig. 2

Finally, the quadrilateral $IA'OC'$ is circumscribed due to the right angles at A' and C' (see Fig. 2). On its circumcircle, we have $\widehat{A'OC'} = 2\angle A'IC' < 180° = \widehat{OC'I}$, hence $180° \ge \widehat{IC'} > \widehat{A'O}$. This means that $IC' > A'O$. Finally, we have $OI \le IA' + A'O < IA' + IC' \le (R' - r) + (r - r') = R' - r' = r'$, as desired.

Solution 2 for G6. Assume the contrary. Then the incenter I should lie in one of triangles AYZ, BXZ, CXY — assume that it lies in $\triangle AYZ$. Let the incircle ω of $\triangle ABC$ touch sides BC, AC at point A_1, B_1 respectively. Without loss of generality, assume that point A_1 lies on segment CX. In this case we will show that $\angle C > 90°$ thus leading to a contradiction.

Note that ω intersects each of the segments XY and YZ at two points; let U, U' and V, V' be the points of intersection of ω with XY and YZ, respectively ($UY > U'Y$, $VY > V'Y$; see Figs. 3 and 4). Note that $60° = \angle XYZ = \frac{1}{2}(\widehat{UV} - \widehat{U'V'}) \le \frac{1}{2}\widehat{UV}$, hence $\widehat{UV} \ge 120°$.

On the other hand, since I lies in $\triangle AYZ$, we get $\widehat{VUV'} < 180°$, hence $\widehat{UA_1U'} \le \widehat{UA_1V'} < 180° - \widehat{UV} \le 60°$.

Now, two cases are possible due to the order of points Y, B_1 on segment AC.

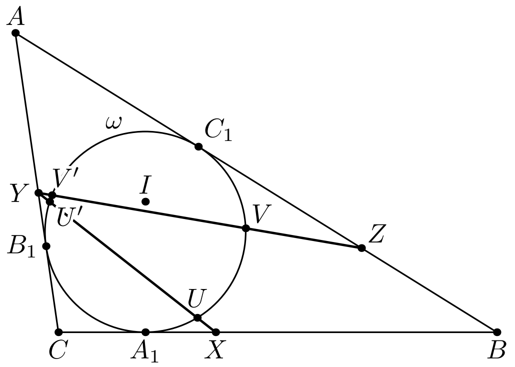


Fig. 3

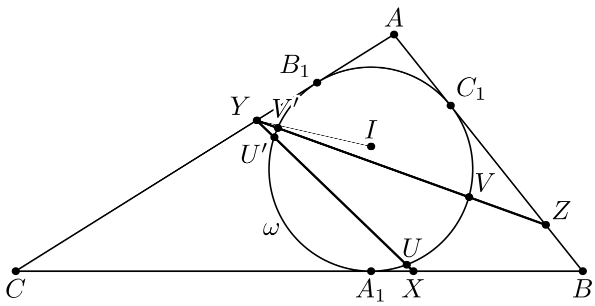


Fig. 4

Case 1. Let point Y lie on the segment AB_1 (see Fig. 3). Then we have $\angle YXC = \frac{1}{2}\left(\widehat{A_1U'} - \widehat{A_1U}\right) \le \frac{1}{2}\widehat{UA_1U'} < 30°$; analogously, we get $\angle XYC \le \frac{1}{2}\widehat{UA_1U'} < 30°$. Therefore, $\angle YCX = 180° - \angle YXC - \angle XYC > 120°$, as desired.

Case 2. Now let point Y lie on the segment CB_1 (see Fig. 4). Analogously, we obtain $\angle YXC < 30°$. Next, $\angle IYX > \angle ZYX = 60°$, but $\angle IYX < \angle IYB_1$, since YB_1 is a tangent and YX is a secant line to circle ω from point Y. Hence, we get $120° < \angle IYB_1 + \angle IYX = \angle B_1YX = \angle YXC + \angle YCX < 30° + \angle YCX$, hence $\angle YCX > 120° - 30° = 90°$, as desired.

Comment. In the same way, one can prove a more general

Claim. Let the vertices X, Y, Z of a triangle XYZ lie respectively on the sides BC, CA, AB of a triangle ABC. Suppose that the incenter of triangle ABC lies outside triangle XYZ, and α is the least angle of $\triangle XYZ$. Then one of the angles of triangle ABC is greater than $3\alpha - 90°$.

Solution for G6′. Assume the contrary. As in Solution 2, we assume that the incenter I of $\triangle ABC$ lies in $\triangle AYZ$, and the tangency point A_1 of ω and BC lies on segment CX. Surely, $\angle YZA \le 180° - \angle YZX = 120°$, hence points I and Y lie on one side of the perpendicular bisector to XY; therefore $IX > IY$. Moreover, ω intersects segment XY at two points, and therefore the projection M of I onto XY lies on the segment XY. In this case, we will prove that $\angle C > 120°$.

Let YK, YL be two tangents from point Y to ω (points K and A_1 lie on one side of XY; if Y lies on ω, we say $K = L = Y$); one of the points K and L is in fact a tangency point B_1 of ω and AC. From symmetry, we have $\angle YIK = \angle YIL$. On the other hand, since $IX > IY$, we get $XM < XY$ which implies $\angle A_1XY < \angle KYX$.

Next, we have $\angle MIY = 90° - \angle IYX < 90° - \angle ZYX = 30°$. Since $IA_1 \perp A_1X$, $IM \perp XY$, $IK \perp YK$ we get $\angle MIA_1 = \angle A_1XY < \angle KYX = \angle MIK$. Finally, we get

$$\begin{aligned}\angle A_1IK < \angle A_1IL &= (\angle A_1IM + \angle MIK) + (\angle KIY + \angle YIL)\\ &< 2\angle MIK + 2\angle KIY = 2\angle MIY < 60°.\end{aligned}$$

Hence, $\angle A_1IB_1 < 60°$, and therefore $\angle ACB = 180° - \angle A_1IB_1 > 120°$, as desired.

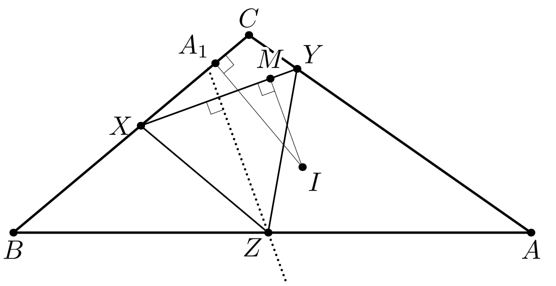


Fig. 5

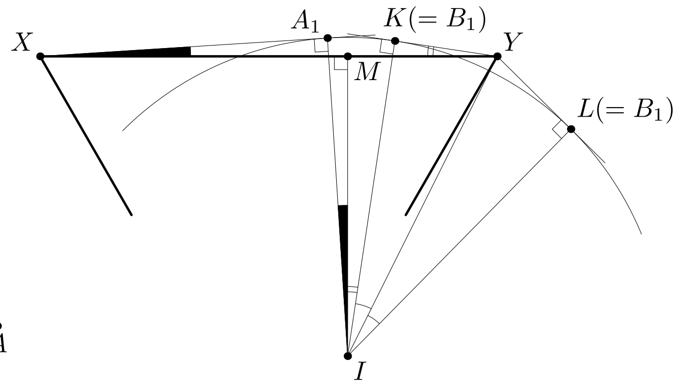


Fig. 6

Comment 1. The estimate claimed in G6′ is sharp. Actually, if $\angle BAC > 120^\circ$, one can consider an equilateral triangle XYZ with $Z = A$, $Y \in AC$, $X \in BC$ (such triangle exists since $\angle ACB < 60^\circ$). It intersects with the angle bisector of $\angle BAC$ only at point A, hence it does not contain I.

Comment 2. As in the previous solution, there is a generalization for an arbitrary triangle XYZ, but here we need some additional condition. The statement reads as follows.

Claim. Let the vertices X, Y, Z of a triangle XYZ lie respectively on the sides BC, CA, AB of a triangle ABC. Suppose that the incenter of triangle ABC lies outside triangle XYZ, α is the least angle of $\triangle XYZ$, and all sides of triangle XYZ are greater than $2r \cot \alpha$, where r is the inradius of $\triangle ABC$. Then one of the angles of triangle ABC is greater than 2α.

The additional condition is needed to verify that $XM > YM$ since it cannot be shown in the original way. Actually, we have $\angle MYI > \alpha$, $IM < r$, hence $YM < r \cot \alpha$. Now, if we have $XY = XM + YM > 2r \cot \alpha$, then surely $XM > YM$.

On the other hand, this additional condition follows easily from the conditions of the original problem. Actually, if $I \in \triangle AYZ$, then the diameter of ω parallel to YZ is contained in $\triangle AYZ$ and is thus shorter than YZ. Hence $YZ > 2r > 2r \cot 60^\circ$.

G7. Three circular arcs γ_1, γ_2, and γ_3 connect the points A and C. These arcs lie in the same half-plane defined by line AC in such a way that arc γ_2 lies between the arcs γ_1 and γ_3. Point B lies on the segment AC. Let h_1, h_2, and h_3 be three rays starting at B, lying in the same half-plane, h_2 being between h_1 and h_3. For $i, j = 1, 2, 3$, denote by V_{ij} the point of intersection of h_i and γ_j (see the Figure below).

Denote by $\widehat{V_{ij}V_{kj}}\widehat{V_{k\ell}V_{i\ell}}$ the curved quadrilateral, whose sides are the segments $V_{ij}V_{i\ell}$, $V_{kj}V_{k\ell}$ and arcs $V_{ij}V_{kj}$ and $V_{i\ell}V_{k\ell}$. We say that this quadrilateral is *circumscribed* if there exists a circle touching these two segments and two arcs.

Prove that if the curved quadrilaterals $\widehat{V_{11}V_{21}}\widehat{V_{22}V_{12}}$, $\widehat{V_{12}V_{22}}\widehat{V_{23}V_{13}}$, $\widehat{V_{21}V_{31}}\widehat{V_{32}V_{22}}$ are circumscribed, then the curved quadrilateral $\widehat{V_{22}V_{32}}\widehat{V_{33}V_{23}}$ is circumscribed, too.

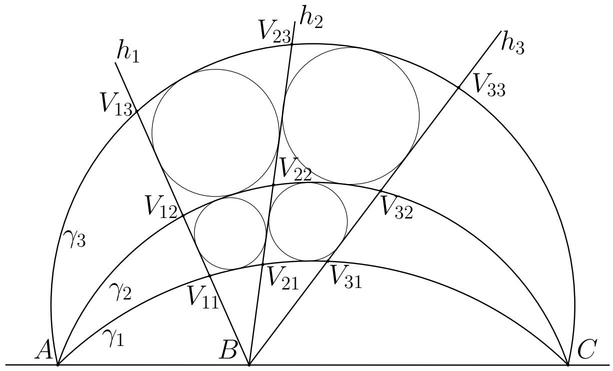


Fig. 1

(Hungary)

Solution. Denote by O_i and R_i the center and the radius of γ_i, respectively. Denote also by H the half-plane defined by AC which contains the whole configuration. For every point P in the half-plane H, denote by $d(P)$ the distance between P and line AC. Furthermore, for any $r > 0$, denote by $\Omega(P, r)$ the circle with center P and radius r.

Lemma 1. For every $1 \le i < j \le 3$, consider those circles $\Omega(P, r)$ in the half-plane H which are tangent to h_i and h_j.

(a) The locus of the centers of these circles is the angle bisector β_{ij} between h_i and h_j.

(b) There is a constant u_{ij} such that $r = u_{ij} \cdot d(P)$ for all such circles.

Proof. Part (a) is obvious. To prove part (b), notice that the circles which are tangent to h_i and h_j are homothetic with the common homothety center B (see Fig. 2). Then part (b) also becomes trivial. □

Lemma 2. For every $1 \le i < j \le 3$, consider those circles $\Omega(P, r)$ in the half-plane H which are externally tangent to γ_i and internally tangent to γ_j.

(a) The locus of the centers of these circles is an ellipse arc ε_{ij} with end-points A and C.

(b) There is a constant v_{ij} such that $r = v_{ij} \cdot d(P)$ for all such circles.

Proof. (a) Notice that the circle $\Omega(P, r)$ is externally tangent to γ_i and internally tangent to γ_j if and only if $O_iP = R_i + r$ and $O_j = R_j - r$. Therefore, for each such circle we have

$$O_iP + O_jP = O_iA + O_jA = O_iC + O_jC = R_i + R_j.$$

Such points lie on an ellipse with foci O_i and O_j; the diameter of this ellipse is $R_i + R_j$, and it passes through the points A and C. Let ε_{ij} be that arc AC of the ellipse which runs inside the half plane H (see Fig. 3.)

This ellipse arc lies between the arcs γ_i and γ_j. Therefore, if some point P lies on ε_{ij}, then $O_iP > R_i$ and $O_jP < R_j$. Now, we choose $r = O_iP - R_i = R_j - O_jP > 0$; then the

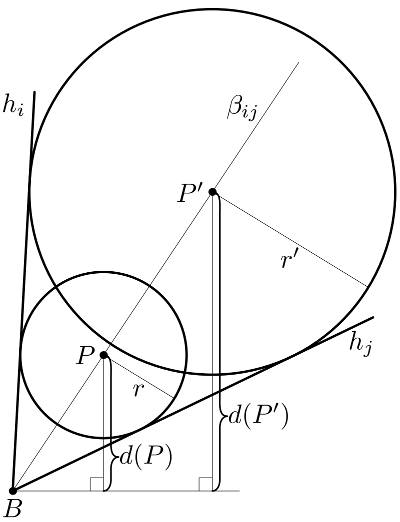


Fig. 2

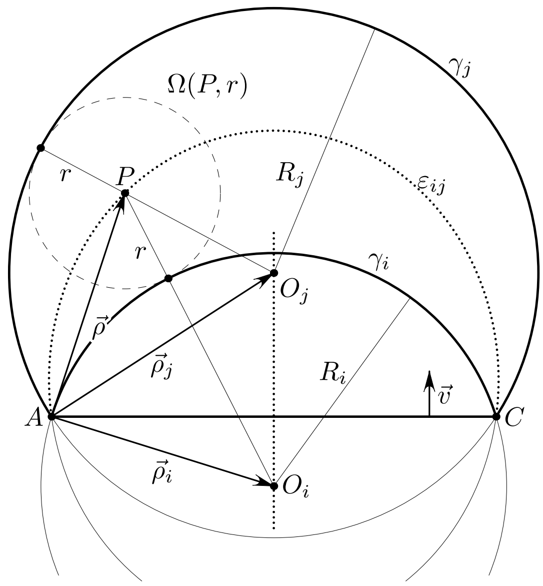


Fig. 3

circle $\Omega(P, r)$ touches γ_i externally and touches γ_j internally, so P belongs to the locus under investigation.

(b) Let $\vec{\rho} = \overrightarrow{AP}$, $\vec{\rho}_i = \overrightarrow{AO_i}$, and $\vec{\rho}_j = \overrightarrow{AO_j}$; let $d_{ij} = O_iO_j$, and let $\vec{v}$ be a unit vector orthogonal to AC and directed toward H. Then we have $|\vec{\rho}_i| = R_i$, $|\vec{\rho}_j| = R_j$, $|\overrightarrow{O_iP}| = |\vec{\rho} - \vec{\rho}_i| = R_i + r$, $|\overrightarrow{O_jP}| = |\vec{\rho} - \vec{\rho}_j| = R_j - r$, hence

$$(\vec{\rho} - \vec{\rho}_i)^2 - (\vec{\rho} - \vec{\rho}_j)^2 = (R_i + r)^2 - (R_j - r)^2,$$
$$(\vec{\rho}_i^{\,2} - \vec{\rho}_j^{\,2}) + 2\vec{\rho} \cdot (\vec{\rho}_j - \vec{\rho}_i) = (R_i^2 - R_j^2) + 2r(R_i + R_j),$$
$$d_{ij} \cdot d(P) = d_{ij}\vec{v} \cdot \vec{\rho} = (\vec{\rho}_j - \vec{\rho}_i) \cdot \vec{\rho} = r(R_i + R_j).$$

Therefore,

$$r = \frac{d_{ij}}{R_i + R_j} \cdot d(P),$$

and the value $v_{ij} = \dfrac{d_{ij}}{R_i + R_j}$ does not depend on P. □

Lemma 3. The curved quadrilateral $\mathcal{Q}_{ij} = V_{i,j}\widehat{V_{i+1,j}}V_{i+1,j+1}\widehat{V_{i,j+1}}$ is circumscribed if and only if $u_{i,i+1} = v_{j,j+1}$.

Proof. First suppose that the curved quadrilateral $\mathcal{Q}_{ij}$ is circumscribed and $\Omega(P, r)$ is its inscribed circle. By Lemma 1 and Lemma 2 we have $r = u_{i,i+1} \cdot d(P)$ and $r = v_{j,j+1} \cdot d(P)$ as well. Hence, $u_{i,i+1} = v_{j,j+1}$.

To prove the opposite direction, suppose $u_{i,i+1} = v_{j,j+1}$. Let P be the intersection of the angle bisector $\beta_{i,i+1}$ and the ellipse arc $\varepsilon_{j,j+1}$. Choose $r = u_{i,i+1} \cdot d(P) = v_{j,j+1} \cdot d(P)$. Then the circle $\Omega(P, r)$ is tangent to the half lines h_i and h_{i+1} by Lemma 1, and it is tangent to the arcs γ_j and γ_{j+1} by Lemma 2. Hence, the curved quadrilateral $\mathcal{Q}_{ij}$ is circumscribed. □

By Lemma 3, the statement of the problem can be reformulated to an obvious fact: If the equalities $u_{12} = v_{12}$, $u_{12} = v_{23}$, and $u_{23} = v_{12}$ hold, then $u_{23} = v_{23}$ holds as well.

Comment 1. Lemma 2(b) (together with the easy Lemma 1(b)) is the key tool in this solution. If one finds this fact, then the solution can be finished in many ways. That is, one can find a circle touching three of h_2, h_3, γ_2, and γ_3, and then prove that it is tangent to the fourth one in either synthetic or analytical way. Both approaches can be successful.

Here we present some discussion about this key Lemma.

1. In the solution above we chose an analytic proof for Lemma 2(b) because we expect that most students will use coordinates or vectors to examine the locus of the centers, and these approaches are less case-sensitive.

Here we outline a synthetic proof. We consider only the case when P does not lie in the line O_iO_j. The other case can be obtained as a limit case, or computed in a direct way.

Let S be the internal homothety center between the circles of γ_i and γ_j, lying on O_iO_j; this point does not depend on P. Let U and V be the points of tangency of circle $\sigma = \Omega(P, r)$ with γ_i and γ_j, respectively (then $r = PU = PV$); in other words, points U and V are the intersection points of rays O_iP, O_jP with arcs γ_i, γ_j respectively (see Fig. 4).

Due to the theorem on three homothety centers (or just to the Menelaus theorem applied to triangle O_iO_jP), the points U, V and S are collinear. Let T be the intersection point of line AC and the common tangent to σ and γ_i at U; then T is the radical center of σ, γ_i and γ_j, hence TV is the common tangent to σ and γ_j.

Let Q be the projection of P onto the line AC. By the right angles, the points U, V and Q lie on the circle with diameter PT. From this fact and the equality $PU = PV$ we get $\angle UQP = \angle UVP = \angle VUP = \angle SUO_i$. Since $O_iS \parallel PQ$, we have $\angle SO_iU = \angle QPU$. Hence, the triangles SO_iU and UPQ are similar and thus $\dfrac{r}{d(P)} = \dfrac{PU}{PQ} = \dfrac{O_iS}{O_iU} = \dfrac{O_iS}{R_i}$; the last expression is constant since S is a constant point. □

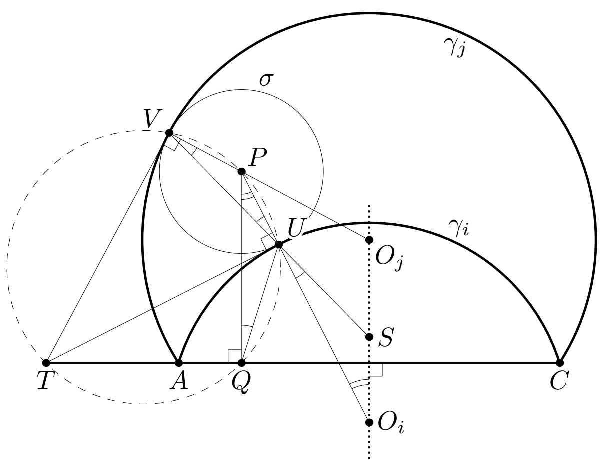


Fig. 4

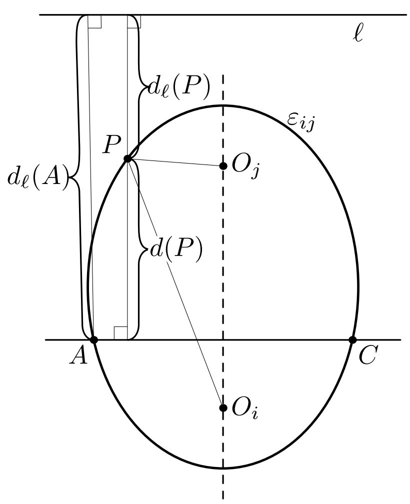


Fig. 5

2. Using some known facts about conics, the same statement can be proved in a very short way. Denote by ℓ the directrix of ellipse of ε_{ij} related to the focus O_j; since ε_{ij} is symmetrical about O_iO_j, we have $\ell \parallel AC$. Recall that for each point $P \in \varepsilon_{ij}$, we have $PO_j = \epsilon \cdot d_\ell(P)$, where $d_\ell(P)$ is the distance from P to ℓ, and ϵ is the eccentricity of ε_{ij} (see Fig. 5).

Now we have

$$r = R_j - (R_j - r) = AO_j - PO_j = \epsilon\big(d_\ell(A) - d_\ell(P)\big) = \epsilon\big(d(P) - d(A)\big) = \epsilon \cdot d(P),$$

and ϵ does not depend on P. □

Comment 2. One can find a spatial interpretations of the problem and the solution.

For every point (x, y) and radius $r > 0$, represent the circle $\Omega\big((x, y), r\big)$ by the point (x, y, r) in space. This point is the apex of the cone with base circle $\Omega\big((x, y), r\big)$ and height r. According to Lemma 1, the circles which are tangent to h_i and h_j correspond to the points of a half line β'_{ij}, starting at B.

Now we translate Lemma 2. Take some $1 \le i < j \le 3$, and consider those circles which are internally tangent to γ_j. It is easy to see that the locus of the points which represent these circles is a subset of a cone, containing γ_j. Similarly, the circles which are externally tangent to γ_i correspond to the points on the extension of another cone, which has its apex on the opposite side of the base plane Π. (See Fig. 6; for this illustration, the z-coordinates were multiplied by 2.)

The two cones are symmetric to each other (they have the same aperture, and their axes are parallel). As is well-known, it follows that the common points of the two cones are co-planar. So the intersection of the two cones is a a conic section — which is an ellipse, according to Lemma 2(a). The points which represent the circles touching γ_i and γ_j is an ellipse arc ε'_{ij} with end-points A and C.

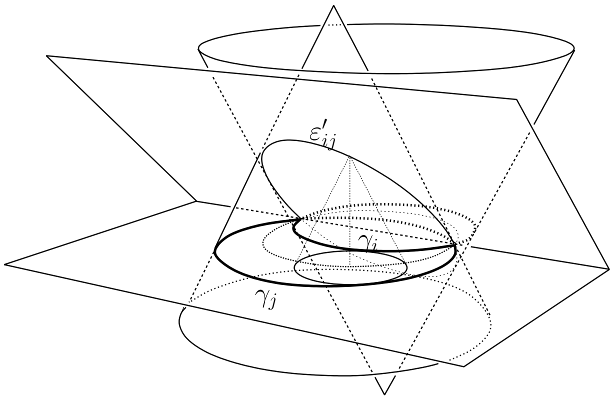


Fig. 6

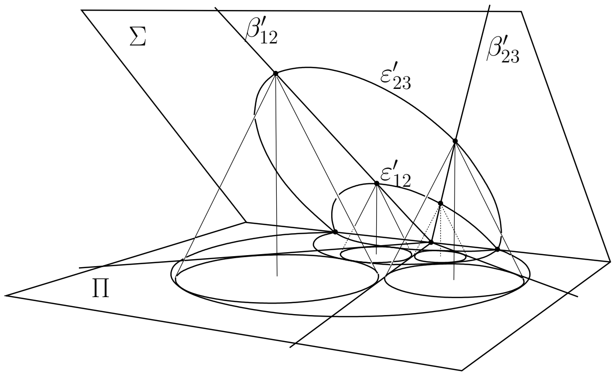
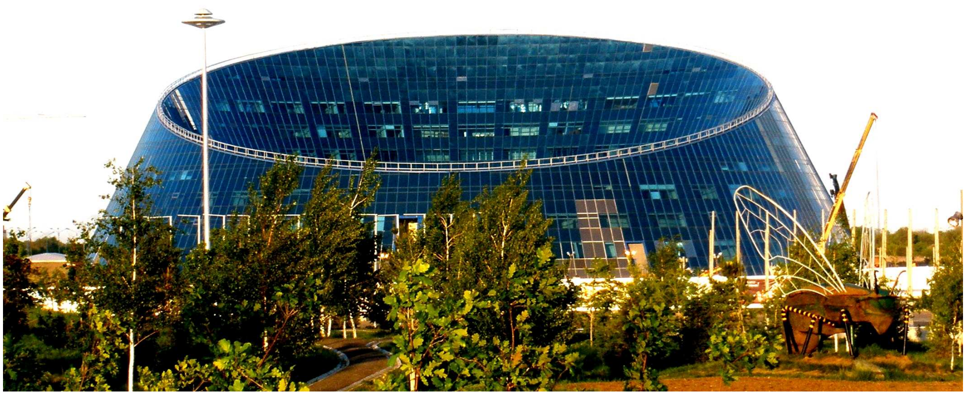


Fig. 7

Thus, the curved quadrilateral $\mathcal{Q}_{ij}$ is circumscribed if and only if $\beta'_{i,i+1}$ and $\varepsilon'_{j,j+1}$ intersect, i.e. if they are coplanar. If three of the four curved quadrilaterals are circumscribed, it means that ε'_{12}, ε'_{23}, β'_{12} and β'_{23} lie in the same plane Σ, and the fourth intersection comes to existence, too (see Fig. 7).



A connection between mathematics and real life:
the Palace of Creativity "Shabyt" ("Inspiration") in Astana

Number Theory

N1. Find the least positive integer n for which there exists a set $\{s_1, s_2, \ldots, s_n\}$ consisting of n distinct positive integers such that

$$\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\ldots\left(1-\frac{1}{s_n}\right)=\frac{51}{2010}.$$

N1′. Same as Problem N1, but the constant $\frac{51}{2010}$ is replaced by $\frac{42}{2010}$.

(Canada)

Answer for Problem N1. $n = 39$.

Solution for Problem N1. Suppose that for some n there exist the desired numbers; we may assume that $s_1 < s_2 < \cdots < s_n$. Surely $s_1 > 1$ since otherwise $1 - \frac{1}{s_1} = 0$. So we have $2 \le s_1 \le s_2 - 1 \le \cdots \le s_n - (n-1)$, hence $s_i \ge i + 1$ for each $i = 1, \ldots, n$. Therefore

$$\begin{aligned}\frac{51}{2010} &= \left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\ldots\left(1-\frac{1}{s_n}\right)\\ &\ge \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{n+1}\right)=\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n}{n+1}=\frac{1}{n+1},\end{aligned}$$

which implies

$$n+1 \ge \frac{2010}{51} = \frac{670}{17} > 39,$$

so $n \ge 39$.

Now we are left to show that $n = 39$ fits. Consider the set $\{2, 3, \ldots, 33, 35, 36, \ldots, 40, 67\}$ which contains exactly 39 numbers. We have

$$\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{32}{33}\cdot\frac{34}{35}\cdots\frac{39}{40}\cdot\frac{66}{67}=\frac{1}{33}\cdot\frac{34}{40}\cdot\frac{66}{67}=\frac{17}{670}=\frac{51}{2010}, \tag{1}$$

hence for $n = 39$ there exists a desired example.

Comment. One can show that the example (1) is unique.

Answer for Problem N1′. $n = 48$.

Solution for Problem N1′. Suppose that for some n there exist the desired numbers. In the same way we obtain that $s_i \ge i + 1$. Moreover, since the denominator of the fraction $\frac{42}{2010} = \frac{7}{335}$ is divisible by 67, some of s_i's should be divisible by 67, so $s_n \ge s_i \ge 67$. This means that

$$\frac{42}{2010} \ge \frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n-1}{n}\cdot\left(1-\frac{1}{67}\right)=\frac{66}{67n},$$

which implies

$$n \geq \frac{2010 \cdot 66}{42 \cdot 67} = \frac{330}{7} > 47,$$

so $n \geq 48$.

Now we are left to show that $n = 48$ fits. Consider the set $\{2, 3, \ldots, 33, 36, 37, \ldots, 50, 67\}$ which contains exactly 48 numbers. We have

$$\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{32}{33} \cdot \frac{35}{36} \cdots \frac{49}{50} \cdot \frac{66}{67} = \frac{1}{33} \cdot \frac{35}{50} \cdot \frac{66}{67} = \frac{7}{335} = \frac{42}{2010},$$

hence for $n = 48$ there exists a desired example.

Comment 1. In this version of the problem, the estimate needs one more step, hence it is a bit harder. On the other hand, the example in this version is not unique. Another example is

$$\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{46}{47} \cdot \frac{66}{67} \cdot \frac{329}{330} = \frac{1}{67} \cdot \frac{66}{330} \cdot \frac{329}{47} = \frac{7}{67 \cdot 5} = \frac{42}{2010}.$$

Comment 2. N1$'$ was the Proposer's formulation of the problem. We propose N1 according to the number of current IMO.

N2. Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2 \cdot 3^n = m\bigl(2^{n+1} - 1\bigr). \tag{1}$$

(Australia)

Answer. $(6, 3)$, $(9, 3)$, $(9, 5)$, $(54, 5)$.

Solution. For fixed values of n, the equation (1) is a simple quadratic equation in m. For $n \le 5$ the solutions are listed in the following table.

case	equation	discriminant	integer roots
$n = 0$	$m^2 - m + 2 = 0$	-7	none
$n = 1$	$m^2 - 3m + 6 = 0$	-15	none
$n = 2$	$m^2 - 7m + 18 = 0$	-23	none
$n = 3$	$m^2 - 15m + 54 = 0$	9	$m = 6$ and $m = 9$
$n = 4$	$m^2 - 31m + 162 = 0$	313	none
$n = 5$	$m^2 - 63m + 486 = 0$	$2025 = 45^2$	$m = 9$ and $m = 54$

We prove that there is no solution for $n \ge 6$.

Suppose that (m, n) satisfies (1) and $n \ge 6$. Since $m \mid 2 \cdot 3^n = m\bigl(2^{n+1} - 1\bigr) - m^2$, we have $m = 3^p$ with some $0 \le p \le n$ or $m = 2 \cdot 3^q$ with some $0 \le q \le n$.

In the first case, let $q = n - p$; then

$$2^{n+1} - 1 = m + \frac{2 \cdot 3^n}{m} = 3^p + 2 \cdot 3^q.$$

In the second case let $p = n - q$. Then

$$2^{n+1} - 1 = m + \frac{2 \cdot 3^n}{m} = 2 \cdot 3^q + 3^p.$$

Hence, in both cases we need to find the nonnegative integer solutions of

$$3^p + 2 \cdot 3^q = 2^{n+1} - 1, \quad p + q = n. \tag{2}$$

Next, we prove bounds for p, q. From (2) we get

$$3^p < 2^{n+1} = 8^{\frac{n+1}{3}} < 9^{\frac{n+1}{3}} = 3^{\frac{2(n+1)}{3}}$$

and

$$2 \cdot 3^q < 2^{n+1} = 2 \cdot 8^{\frac{n}{3}} < 2 \cdot 9^{\frac{n}{3}} = 2 \cdot 3^{\frac{2n}{3}} < 2 \cdot 3^{\frac{2(n+1)}{3}},$$

so $p, q < \frac{2(n+1)}{3}$. Combining these inequalities with $p + q = n$, we obtain

$$\frac{n-2}{3} < p, q < \frac{2(n+1)}{3}. \tag{3}$$

Now let $h = \min(p, q)$. By (3) we have $h > \frac{n-2}{3}$; in particular, we have $h > 1$. On the left-hand side of (2), both terms are divisible by 3^h, therefore $9 \mid 3^h \mid 2^{n+1} - 1$. It is easy check that $\operatorname{ord}_9(2) = 6$, so $9 \mid 2^{n+1} - 1$ if and only if $6 \mid n + 1$. Therefore, $n + 1 = 6r$ for some positive integer r, and we can write

$$2^{n+1} - 1 = 4^{3r} - 1 = (4^{2r} + 4^r + 1)(2^r - 1)(2^r + 1). \tag{4}$$

Notice that the factor $4^{2r} + 4^r + 1 = (4^r - 1)^2 + 3 \cdot 4^r$ is divisible by 3, but it is never divisible by 9. The other two factors in (4), $2^r - 1$ and $2^r + 1$ are coprime: both are odd and their difference is 2. Since the whole product is divisible by 3^h, we have either $3^{h-1} \mid 2^r - 1$ or $3^{h-1} \mid 2^r + 1$. In any case, we have $3^{h-1} \leq 2^r + 1$. Then

$$3^{h-1} \leq 2^r + 1 \leq 3^r = 3^{\frac{n+1}{6}},$$
$$\frac{n-2}{3} - 1 < h - 1 \leq \frac{n+1}{6},$$
$$n < 11.$$

But this is impossible since we assumed $n \geq 6$, and we proved $6 \mid n + 1$.

N3. Find the smallest number n such that there exist polynomials $f_1, f_2, \dots, f_n$ with rational coefficients satisfying

$$x^2+7=f_1(x)^2+f_2(x)^2+\cdots+f_n(x)^2.$$

(Poland)

Answer. The smallest n is 5.

Solution 1. The equality $x^2+7=x^2+2^2+1^2+1^2+1^2$ shows that $n \le 5$. It remains to show that x^2+7 is not a sum of four (or less) squares of polynomials with rational coefficients.

Suppose by way of contradiction that $x^2+7=f_1(x)^2+f_2(x)^2+f_3(x)^2+f_4(x)^2$, where the coefficients of polynomials f_1, f_2, f_3 and f_4 are rational (some of these polynomials may be zero).

Clearly, the degrees of f_1, f_2, f_3 and f_4 are at most 1. Thus $f_i(x)=a_ix+b_i$ for $i=1,2,3,4$ and some rationals a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4. It follows that $x^2+7=\sum_{i=1}^4(a_ix+b_i)^2$ and hence

$$\sum_{i=1}^4 a_i^2=1, \quad \sum_{i=1}^4 a_ib_i=0, \quad \sum_{i=1}^4 b_i^2=7. \tag{1}$$

Let $p_i=a_i+b_i$ and $q_i=a_i-b_i$ for $i=1,2,3,4$. Then

$$\sum_{i=1}^4 p_i^2=\sum_{i=1}^4 a_i^2+2\sum_{i=1}^4 a_ib_i+\sum_{i=1}^4 b_i^2=8,$$
$$\sum_{i=1}^4 q_i^2=\sum_{i=1}^4 a_i^2-2\sum_{i=1}^4 a_ib_i+\sum_{i=1}^4 b_i^2=8$$
$$\text{and} \quad \sum_{i=1}^4 p_iq_i=\sum_{i=1}^4 a_i^2-\sum_{i=1}^4 b_i^2=-6,$$

which means that there exist a solution in integers x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4 and $m>0$ of the system of equations

$$\text{(i)} \ \sum_{i=1}^4 x_i^2=8m^2, \qquad \text{(ii)} \ \sum_{i=1}^4 y_i^2=8m^2, \qquad \text{(iii)} \ \sum_{i=1}^4 x_iy_i=-6m^2.$$

We will show that such a solution does not exist.

Assume the contrary and consider a solution with minimal m. Note that if an integer x is odd then $x^2 \equiv 1 \pmod 8$. Otherwise (i.e., if x is even) we have $x^2 \equiv 0 \pmod 8$ or $x^2 \equiv 4 \pmod 8$. Hence, by (i), we get that x_1, x_2, x_3 and x_4 are even. Similarly, by (ii), we get that y_1, y_2, y_3 and y_4 are even. Thus the LHS of (iii) is divisible by 4 and m is also even. It follows that $(\frac{x_1}{2}, \frac{y_1}{2}, \frac{x_2}{2}, \frac{y_2}{2}, \frac{x_3}{2}, \frac{y_3}{2}, \frac{x_4}{2}, \frac{y_4}{2}, \frac{m}{2})$ is a solution of the system of equations (i), (ii) and (iii), which contradicts the minimality of m.

Solution 2. We prove that $n \le 4$ is impossible. Define the numbers a_i, b_i for $i=1,2,3,4$ as in the previous solution.

By Euler's identity we have

$$(a_1^2+a_2^2+a_3^2+a_4^2)(b_1^2+b_2^2+b_3^2+b_4^2)=(a_1b_1+a_2b_2+a_3b_3+a_4b_4)^2+(a_1b_2-a_2b_1+a_3b_4-a_4b_3)^2$$
$$+(a_1b_3-a_3b_1+a_4b_2-a_2b_4)^2+(a_1b_4-a_4b_1+a_2b_3-a_3b_2)^2.$$

So, using the relations (1) from the Solution 1 we get that

$$7 = \left(\frac{m_1}{m}\right)^2 + \left(\frac{m_2}{m}\right)^2 + \left(\frac{m_3}{m}\right)^2, \qquad (2)$$

where

$$\frac{m_1}{m} = a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3,$$
$$\frac{m_2}{m} = a_1b_3 - a_3b_1 + a_4b_2 - a_2b_4,$$
$$\frac{m_3}{m} = a_1b_4 - a_4b_1 + a_2b_3 - a_3b_2$$

and $m_1, m_2, m_3 \in \mathbb{Z}, m \in \mathbb{N}$.

Let m be a minimum positive integer number for which (2) holds. Then

$$8m^2 = m_1^2 + m_2^2 + m_3^2 + m^2.$$

As in the previous solution, we get that m_1, m_2, m_3, m are all even numbers. Then $\left(\frac{m_1}{2}, \frac{m_2}{2}, \frac{m_3}{2}, \frac{m}{2}\right)$ is also a solution of (2) which contradicts the minimality of m. So, we have $n \geq 5$. The example with $n = 5$ is already shown in Solution 1.

N4. Let a, b be integers, and let $P(x) = ax^3 + bx$. For any positive integer n we say that the pair (a, b) is *n-good* if $n \mid P(m) - P(k)$ implies $n \mid m - k$ for all integers m, k. We say that (a, b) is *very good* if (a, b) is n-good for infinitely many positive integers n.

(a) Find a pair (a, b) which is 51-good, but not very good.

(b) Show that all 2010-good pairs are very good.

(Turkey)

Solution. (a) We show that the pair $(1, -51^2)$ is good but not very good. Let $P(x) = x^3 - 51^2 x$. Since $P(51) = P(0)$, the pair $(1, -51^2)$ is not n-good for any positive integer that does not divide 51. Therefore, $(1, -51^2)$ is not very good.

On the other hand, if $P(m) \equiv P(k) \pmod{51}$, then $m^3 \equiv k^3 \pmod{51}$. By Fermat's theorem, from this we obtain

$$m \equiv m^3 \equiv k^3 \equiv k \pmod 3 \quad \text{and} \quad m \equiv m^{33} \equiv k^{33} \equiv k \pmod{17}.$$

Hence we have $m \equiv k \pmod{51}$. Therefore $(1, -51^2)$ is 51-good.

(b) We will show that if a pair (a, b) is 2010-good then (a, b) is 67^i-good for all positive integer i.

Claim 1. If (a, b) is 2010-good then (a, b) is 67-good.

Proof. Assume that $P(m) = P(k) \pmod{67}$. Since 67 and 30 are coprime, there exist integers m' and k' such that $k' \equiv k \pmod{67}$, $k' \equiv 0 \pmod{30}$, and $m' \equiv m \pmod{67}$, $m' \equiv 0 \pmod{30}$. Then we have $P(m') \equiv P(0) \equiv P(k') \pmod{30}$ and $P(m') \equiv P(m) \equiv P(k) \equiv P(k') \pmod{67}$, hence $P(m') \equiv P(k') \pmod{2010}$. This implies $m' \equiv k' \pmod{2010}$ as (a, b) is 2010-good. It follows that $m \equiv m' \equiv k' \equiv k \pmod{67}$. Therefore, (a, b) is 67-good. □

Claim 2. If (a, b) is 67-good then $67 \mid a$.

Proof. Suppose that $67 \nmid a$. Consider the sets $\{at^2 \pmod{67} : 0 \le t \le 33\}$ and $\{-3as^2 - b \mod 67 : 0 \le s \le 33\}$. Since $a \not\equiv 0 \pmod{67}$, each of these sets has 34 elements. Hence they have at least one element in common. If $at^2 \equiv -3as^2 - b \pmod{67}$ then for $m = t \pm s$, $k = \mp 2s$ we have

$$\begin{aligned} P(m) - P(k) = a(m^3 - k^3) + b(m - k) &= (m - k)\big(a(m^2 + mk + k^2) + b\big) \\ &= (t \pm 3s)(at^2 + 3as^2 + b) \equiv 0 \pmod{67}. \end{aligned}$$

Since (a, b) is 67-good, we must have $m \equiv k \pmod{67}$ in both cases, that is, $t \equiv 3s \pmod{67}$ and $t \equiv -3s \pmod{67}$. This means $t \equiv s \equiv 0 \pmod{67}$ and $b \equiv -3as^2 - at^2 \equiv 0 \pmod{67}$. But then $67 \mid P(7) - P(2) = 67 \cdot 5a + 5b$ and $67 \nmid 7 - 2$, contradicting that (a, b) is 67-good. □

Claim 3. If (a, b) is 2010-good then (a, b) is 67^i-good all $i \ge 1$.

Proof. By Claim 2, we have $67 \mid a$. If $67 \mid b$, then $P(x) \equiv P(0) \pmod{67}$ for all x, contradicting that (a, b) is 67-good. Hence, $67 \nmid b$.

Suppose that $67^i \mid P(m) - P(k) = (m - k)\big(a(m^2 + mk + k^2) + b\big)$. Since $67 \mid a$ and $67 \nmid b$, the second factor $a(m^2 + mk + k^2) + b$ is coprime to 67 and hence $67^i \mid m - k$. Therefore, (a, b) is 67^i-good. □

Comment 1. In the proof of Claim 2, the following reasoning can also be used. Since 3 is not a quadratic residue modulo 67, either $au^2 \equiv -b \pmod{67}$ or $3av^2 \equiv -b \pmod{67}$ has a solution. The settings $(m, k) = (u, 0)$ in the first case and $(m, k) = (v, -2v)$ in the second case lead to $b \equiv 0 \pmod{67}$.

Comment 2. The pair $(67, 30)$ is n-good if and only if $n = d \cdot 67^i$, where $d \mid 30$ and $i \ge 0$. It shows that in part (b), one should deal with the large powers of 67 to reach the solution. The key property of number 67 is that it has the form $3k + 1$, so there exists a nontrivial cubic root of unity modulo 67.

N5. Let $\mathbb{N}$ be the set of all positive integers. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that the number $\big(f(m)+n\big)\big(m+f(n)\big)$ is a square for all $m, n \in \mathbb{N}$.

(U.S.A.)

Answer. All functions of the form $f(n) = n + c$, where $c \in \mathbb{N} \cup \{0\}$.

Solution. First, it is clear that all functions of the form $f(n) = n + c$ with a constant nonnegative integer c satisfy the problem conditions since $\big(f(m)+n\big)\big(f(n)+m\big) = (n+m+c)^2$ is a square.

We are left to prove that there are no other functions. We start with the following

Lemma. Suppose that $p \mid f(k) - f(\ell)$ for some prime p and positive integers k, ℓ. Then $p \mid k - \ell$.

Proof. Suppose first that $p^2 \mid f(k) - f(\ell)$, so $f(\ell) = f(k) + p^2 a$ for some integer a. Take some positive integer $D > \max\{f(k), f(\ell)\}$ which is not divisible by p and set $n = pD - f(k)$. Then the positive numbers $n + f(k) = pD$ and $n + f(\ell) = pD + \big(f(\ell) - f(k)\big) = p(D + pa)$ are both divisible by p but not by p^2. Now, applying the problem conditions, we get that both the numbers $\big(f(k)+n\big)\big(f(n)+k\big)$ and $\big(f(\ell)+n\big)\big(f(n)+\ell\big)$ are squares divisible by p (and thus by p^2); this means that the multipliers $f(n)+k$ and $f(n)+\ell$ are also divisible by p, therefore $p \mid \big(f(n)+k\big) - \big(f(n)+\ell\big) = k - \ell$ as well.

On the other hand, if $f(k) - f(\ell)$ is divisible by p but not by p^2, then choose the same number D and set $n = p^3 D - f(k)$. Then the positive numbers $f(k) + n = p^3 D$ and $f(\ell) + n = p^3 D + \big(f(\ell) - f(k)\big)$ are respectively divisible by p^3 (but not by p^4) and by p (but not by p^2). Hence in analogous way we obtain that the numbers $f(n)+k$ and $f(n)+\ell$ are divisible by p, therefore $p \mid \big(f(n)+k\big) - \big(f(n)+\ell\big) = k - \ell$. □

We turn to the problem. First, suppose that $f(k) = f(\ell)$ for some $k, \ell \in \mathbb{N}$. Then by Lemma we have that $k - \ell$ is divisible by every prime number, so $k - \ell = 0$, or $k = \ell$. Therefore, the function f is injective.

Next, consider the numbers $f(k)$ and $f(k+1)$. Since the number $(k+1) - k = 1$ has no prime divisors, by Lemma the same holds for $f(k+1) - f(k)$; thus $|f(k+1) - f(k)| = 1$.

Now, let $f(2) - f(1) = q$, $|q| = 1$. Then we prove by induction that $f(n) = f(1) + q(n-1)$. The base for $n = 1, 2$ holds by the definition of q. For the step, if $n > 1$ we have $f(n+1) = f(n) \pm q = f(1) + q(n-1) \pm q$. Since $f(n) \neq f(n-2) = f(1) + q(n-2)$, we get $f(n) = f(1) + qn$, as desired.

Finally, we have $f(n) = f(1) + q(n-1)$. Then q cannot be -1 since otherwise for $n \geq f(1)+1$ we have $f(n) \leq 0$ which is impossible. Hence $q = 1$ and $f(n) = (f(1)-1) + n$ for each $n \in \mathbb{N}$, and $f(1) - 1 \geq 0$, as desired.

N6. The rows and columns of a $2^n \times 2^n$ table are numbered from 0 to $2^n - 1$. The cells of the table have been colored with the following property being satisfied: for each $0 \le i, j \le 2^n - 1$, the jth cell in the ith row and the $(i+j)$th cell in the jth row have the same color. (The indices of the cells in a row are considered modulo 2^n.)

Prove that the maximal possible number of colors is 2^n.

(Iran)

Solution. Throughout the solution we denote the cells of the table by coordinate pairs; (i, j) refers to the jth cell in the ith row.

Consider the directed graph, whose vertices are the cells of the board, and the edges are the arrows $(i, j) \to (j, i + j)$ for all $0 \le i, j \le 2^n - 1$. From each vertex (i, j), exactly one edge passes (to $(j, i + j \mod 2^n)$); conversely, to each cell (j, k) exactly one edge is directed (from the cell $(k - j \mod 2^n, j)$). Hence, the graph splits into cycles.

Now, in any coloring considered, the vertices of each cycle should have the same color by the problem condition. On the other hand, if each cycle has its own color, the obtained coloring obviously satisfies the problem conditions. Thus, the maximal possible number of colors is the same as the number of cycles, and we have to prove that this number is 2^n.

Next, consider any cycle $(i_1, j_1), (i_2, j_2), \dots$; we will describe it in other terms. Define a sequence $(a_0, a_1, \dots)$ by the relations $a_0 = i_1$, $a_1 = j_1$, $a_{n+1} = a_n + a_{n-1}$ for all $n \ge 1$ (we say that such a sequence is *a Fibonacci-type sequence*). Then an obvious induction shows that $i_k \equiv a_{k-1} \pmod{2^n}$, $j_k \equiv a_k \pmod{2^n}$. Hence we need to investigate the behavior of Fibonacci-type sequences modulo 2^n.

Denote by $F_0, F_1, \dots$ the Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \ge 0$. We also set $F_{-1} = 1$ according to the recurrence relation.

For every positive integer m, denote by $\nu(m)$ the exponent of 2 in the prime factorization of m, i.e. for which $2^{\nu(m)} \mid m$ but $2^{\nu(m)+1} \nmid m$.

Lemma 1. For every Fibonacci-type sequence $a_0, a_1, a_2, \dots$, and every $k \ge 0$, we have $a_k = F_{k-1}a_0 + F_k a_1$.

Proof. Apply induction on k. The base cases $k = 0, 1$ are trivial. For the step, from the induction hypothesis we get

$$a_{k+1} = a_k + a_{k-1} = (F_{k-1}a_0 + F_k a_1) + (F_{k-2}a_0 + F_{k-1}a_1) = F_k a_0 + F_{k+1}a_1. \qquad \square$$

Lemma 2. For every $m \ge 3$,

(a) we have $\nu(F_{3\cdot 2^{m-2}}) = m$;

(b) $d = 3 \cdot 2^{m-2}$ is the least positive index for which $2^m \mid F_d$;

(c) $F_{3\cdot 2^{m-2}+1} \equiv 1 + 2^{m-1} \pmod{2^m}$.

Proof. Apply induction on m. In the base case $m = 3$ we have $\nu(F_{3\cdot 2^{m-2}}) = F_6 = 8$, so $\nu(F_{3\cdot 2^{m-2}}) = \nu(8) = 3$, the preceding Fibonacci-numbers are not divisible by 8, and indeed $F_{3\cdot 2^{m-2}+1} = F_7 = 13 \equiv 1 + 4 \pmod 8$.

Now suppose that $m > 3$ and let $k = 3 \cdot 2^{m-3}$. By applying Lemma 1 to the Fibonacci-type sequence $F_k, F_{k+1}, \dots$ we get

$$F_{2k} = F_{k-1}F_k + F_k F_{k+1} = (F_{k+1} - F_k)F_k + F_{k+1}F_k = 2F_{k+1}F_k - F_k^2,$$
$$F_{2k+1} = F_k \cdot F_k + F_{k+1} \cdot F_{k+1} = F_k^2 + F_{k+1}^2.$$

By the induction hypothesis, $\nu(F_k) = m - 1$, and F_{k+1} is odd. Therefore we get $\nu(F_k^2) = 2(m-1) > (m-1) + 1 = \nu(2F_k F_{k+1})$, which implies $\nu(F_{2k}) = m$, establishing statement (a).

Moreover, since $F_{k+1} = 1 + 2^{m-2} + a2^{m-1}$ for some integer a, we get

$$F_{2k+1} = F_k^2 + F_{k+1}^2 \equiv 0 + (1 + 2^{m-2} + a2^{m-1})^2 \equiv 1 + 2^{m-1} \pmod{2^m},$$

as desired in statement (c).

We are left to prove that $2^m \nmid F_\ell$ for $\ell < 2k$. Assume the contrary. Since $2^{m-1} \mid F_\ell$, from the induction hypothesis it follows that $\ell > k$. But then we have $F_\ell = F_{k-1}F_{\ell-k} + F_k F_{\ell-k+1}$, where the second summand is divisible by 2^{m-1} but the first one is not (since F_{k-1} is odd and $\ell - k < k$). Hence the sum is not divisible even by 2^{m-1}. A contradiction. □

Now, for every pair of integers $(a, b) \neq (0, 0)$, let $\mu(a, b) = \min\{\nu(a), \nu(b)\}$. By an obvious induction, for every Fibonacci-type sequence $A = (a_0, a_1, \dots)$ we have $\mu(a_0, a_1) = \mu(a_1, a_2) = \dots$; denote this common value by $\mu(A)$. Also denote by $p_n(A)$ the period of this sequence modulo 2^n, that is, the least $p > 0$ such that $a_{k+p} \equiv a_k \pmod{2^n}$ for all $k \geq 0$.

Lemma 3. Let $A = (a_0, a_1, \dots)$ be a Fibonacci-type sequence such that $\mu(A) = k < n$. Then $p_n(A) = 3 \cdot 2^{n-1-k}$.

Proof. First, we note that the sequence $(a_0, a_1, \dots)$ has period p modulo 2^n if and only if the sequence $(a_0/2^k, a_1/2^k, \dots)$ has period p modulo 2^{n-k}. Hence, passing to this sequence we can assume that $k = 0$.

We prove the statement by induction on n. It is easy to see that for $n = 1, 2$ the claim is true; actually, each Fibonacci-type sequence A with $\mu(A) = 0$ behaves as $0, 1, 1, 0, 1, 1, \dots$ modulo 2, and as $0, 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, \dots$ modulo 4 (all pairs of residues from which at least one is odd appear as a pair of consecutive terms in this sequence).

Now suppose that $n \geq 3$ and consider an arbitrary Fibonacci-type sequence $A = (a_0, a_1, \dots)$ with $\mu(A) = 0$. Obviously we should have $p_{n-1}(A) \mid p_n(A)$, or, using the induction hypothesis, $s = 3 \cdot 2^{n-2} \mid p_n(A)$. Next, we may suppose that a_0 is even; hence a_1 is odd, and $a_0 = 2b_0$, $a_1 = 2b_1 + 1$ for some integers b_0, b_1.

Consider the Fibonacci-type sequence $B = (b_0, b_1, \dots)$ starting with (b_0, b_1). Since $a_0 = 2b_0 + F_0$, $a_1 = 2b_1 + F_1$, by an easy induction we get $a_k = 2b_k + F_k$ for all $k \geq 0$. By the induction hypothesis, we have $p_{n-1}(B) \mid s$, hence the sequence $(2b_0, 2b_1, \dots)$ is s-periodic modulo 2^n. On the other hand, by Lemma 2 we have $F_{s+1} \equiv 1 + 2^{n-1} \pmod{2^n}$, $F_{2s} \equiv 0 \pmod{2^n}$, $F_{2s+1} \equiv 1 \pmod{2^n}$, hence

$$a_{s+1} = 2b_{s+1} + F_{s+1} \equiv 2b_1 + 1 + 2^{n-1} \not\equiv 2b_1 + 1 = a_1 \pmod{2^n},$$
$$a_{2s} = 2b_{2s} + F_{2s} \equiv 2b_0 + 0 = a_0 \pmod{2^n},$$
$$a_{2s+1} = 2b_{2s+1} + F_{2s+1} \equiv 2b_1 + 1 = a_1 \pmod{2^n}.$$

The first line means that A is not s-periodic, while the other two provide that $a_{2s} \equiv a_0$, $a_{2s+1} \equiv a_1$ and hence $a_{2s+t} \equiv a_t$ for all $t \geq 0$. Hence $s \mid p_n(A) \mid 2s$ and $p_n(A) \neq s$, which means that $p_n(A) = 2s$, as desired. □

Finally, Lemma 3 provides a straightforward method of counting the number of cycles. Actually, take any number $0 \leq k \leq n - 1$ and consider all the cells (i, j) with $\mu(i, j) = k$. The total number of such cells is $2^{2(n-k)} - 2^{2(n-k-1)} = 3 \cdot 2^{2n-2k-2}$. On the other hand, they are split into cycles, and by Lemma 3 the length of each cycle is $3 \cdot 2^{n-1-k}$. Hence the number of cycles consisting of these cells is exactly $\dfrac{3 \cdot 2^{2n-2k-2}}{3 \cdot 2^{n-1-k}} = 2^{n-k-1}$. Finally, there is only one cell $(0, 0)$ which is not mentioned in the previous computation, and it forms a separate cycle. So the total number of cycles is

$$1 + \sum_{k=0}^{n-1} 2^{n-1-k} = 1 + (1 + 2 + 4 + \dots + 2^{n-1}) = 2^n.$$

Comment. We outline a different proof for the essential part of Lemma 3. That is, we assume that $k = 0$ and show that in this case the period of (a_i) modulo 2^n coincides with the period of the Fibonacci numbers modulo 2^n; then the proof can be finished by the arguments from Lemma 2..

Note that p is a (not necessarily minimal) period of the sequence (a_i) modulo 2^n if and only if we have $a_0 \equiv a_p \pmod{2^n}$, $a_1 \equiv a_{p+1} \pmod{2^n}$, that is,

$$\begin{aligned} a_0 &\equiv a_p \equiv F_{p-1}a_0 + F_p a_1 = F_p(a_1 - a_0) + F_{p+1}a_0 \pmod{2^n}, \\ a_1 &\equiv a_{p+1} = F_p a_0 + F_{p+1}a_1 \pmod{2^n}. \end{aligned} \tag{1}$$

Now, If p is a period of (F_i) then we have $F_p \equiv F_0 = 0 \pmod{2^n}$ and $F_{p+1} \equiv F_1 = 1 \pmod{2^n}$, which by (1) implies that p is a period of (a_i) as well.

Conversely, suppose that p is a period of (a_i). Combining the relations of (1) we get

$$\begin{aligned} 0 = a_1 \cdot a_0 - a_0 \cdot a_1 &\equiv a_1\bigl(F_p(a_1 - a_0) + F_{p+1}a_0\bigr) - a_0(F_p a_0 + F_{p+1}a_1) \\ &= F_p(a_1^2 - a_1a_0 - a_0^2) \pmod{2^n}, \\ a_1^2 - a_1a_0 - a_0^2 = (a_1 - a_0)a_1 - a_0 \cdot a_0 &\equiv (a_1 - a_0)(F_p a_0 + F_{p+1}a_1) - a_0\bigl(F_p(a_1 - a_0) + F_{p+1}a_0\bigr) \\ &= F_{p+1}(a_1^2 - a_1a_0 - a_0^2) \pmod{2^n}. \end{aligned}$$

Since at least one of the numbers a_0, a_1 is odd, the number $a_1^2 - a_1a_0 - a_0^2$ is odd as well. Therefore the previous relations are equivalent with $F_p \equiv 0 \pmod{2^n}$ and $F_{p+1} \equiv 1 \pmod{2^n}$, which means exactly that p is a period of $(F_0, F_1, \dots)$ modulo 2^n.

So, the sets of periods of (a_i) and (F_i) coincide, and hence the minimal periods coincide as well.